OPTIMIZING INFORMATION FRESHNESS IN RANDOM ACCESS CHANNELS

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ABSTRACT

OPTIMIZING INFORMATION FRESHNESS IN RANDOM ACCESS CHANNELS

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In this work, a number of transmission strategies aimed at optimizing information freshness in random access channels are developed and studied. Threshold-ALOHA, an age-aware modification of slotted ALOHA, suggests a fixed age threshold on the terminals before they can become active and attempt transmissions with a constant probability. Threshold ALOHA nearly halves the average Age of Information (AoI) whilst the loss of throughput compared to slotted ALOHA is less than one percent. Mumista, multiple mini slotted threshold Aloha, is a further iteration of threshold Aloha that introduces mini slots before each data slot to enable a reservation based mechanism and improve throughput. The set of parameters that achieve the optimal throughput has been explicitly derived. Under ideal conditions, Mumista can approach theoretical limits of throughput and average age of information as closely as desired. Finally, we investigate the optimality of the threshold policy in a wireless energy transfer setting with a Gilbert-Elliott channel between a single transmitter and receiver pair. We obtain the optimal parameters in closed form.

Keywords: slotted aloha, threshold, age of information, information freshness, mumista, random access, wireless energy transfer, gilbert-elliott, age threshold

ÖZ

RASTGELE ERİŞİM KANALLARINDA BİLGİ TAZELİĞİNİ ENİYİLEMEK

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Bu çalışmada rastgele erişim kanallarında bilgi tazeliğini eniyileştiren birtakım paket gönderim stratejileri geliştirilmiş ve çalışılmıştır. Eşikli Aloha, dilimli Aloha'nın yaş farkında bir varyantı, terminallerin aktif olup sabit bir olasılıkla gönderim yapabilmesinden önce bir yaş eşiğinin varlığını önerir. Eşikli Aloha dilimli Aloha'ya kıyasla kanal verimliliğinden yüzden birden az bir kayıpla bilgi yaşını neredeyse yarıya indirir. Mumista, çoklu mini dilimli eşikli Aloha, eşikli Aloha'nın ileri bir iterasyonu olarak her bir veri diliminden önce mini dilimler kullanılmasını, bu sayede rezervasyon temelli bir mekanizmanın kullanılmasını ve kanal verimliliğinin iyileştirilmesini öne sürer. Mümkün olan en iyi kanal verimliliğini elde eden parametreler açıkça elde edilmiştir. İdeal koşullar altında Mumista kanal verimliliğinin ve ortalama bilgi yaşının teorik sınırlarına istenildiği kadar yaklaşabilir. Son olarak, bir kablosuz enerji transferi senaryosunda bir alıcı-iletici çifti arasındaki Gilbert-Elliott kanalı üstünde eşikli bir protokolün optimalitesi incelenmiştir. İdeal parametreler kapalı formda elde edinilmiştir.

Anahtar Kelimeler: dilimli aloha, eşik, bilgi yaşı, bilgi tazeliği, rastgele erişim, kablosuz enerji transferi, gilbert-elliott, yaş eşiği

"Perfect is the enemy of good." - Voltaire

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AoI	Age of Information
TA	Threshold ALOHA
MC	Markov Chain
SAT	Stationary Age Thinning
DTMC	Discrete Time Markov Chain
MUMISTA	Multiple mini slotted threshold Aloha
WET	Wireless Energy Transfer
ARQ	Automatic Repeat Request
HARQ	Hybrid Automatic Repeat Request
IID	Independent and Identically Distributed
CMDP	Constrained Markov Decision Process

CHAPTER 1

INTRODUCTION

Age of Information (AoI) emerged a decade ago [2, 3] as a metric facilitating the characterization and control of information freshness in status-update based networked systems, including Internet-of-Things (IoT) and Machine-type Communications scenarios. Many classical networking formulations have since been revisited from an AoI analysis and optimization perspective [4, 5, 6, 7, 8, 9, 10]. The addressing of *random access* with an AoI objective is relatively new [11], particularly motivated by applications such as industrial automation, networked control systems, environmental monitoring, health and activity sensing, where multiple sensor nodes send updates of sensed data a common access point on a shared channel.

A series of recent works [1, 11, 12, 13, 14] studied basic abstractions that capture the essence of information aging in this random access environment: (1) time is slotted and nodes are synchronized to the slot timing, (2) concurrent transmissions result in packet loss, (3) nodes make distributed transmission decisions, (4) the longer it takes a node to successfully transmit a packet, the more its corresponding data flow ages. These four are the essential assumptions underlying the problem analyzed in this paper.

As a consequence of these assumptions, in order to keep the time-average age in the network under control, the distributed decision mechanism needs to strike a balance between each node attempting transmission sufficiently often, and more than one transmission attempts at a time being unlikely. This problem is related to the classical problem of distributed stabilization of slotted ALOHA (see, *e.g.*, [15]), revisited here through the lens of AoI, which is a fundamentally different performance objective. Throughput optimality and age optimality in channel access scheduling often do not coincide [8]- a throughput optimal mechanism can be arbitrarily poor in terms of average AoI, however, age optimality requires high throughput, and is often attained at an operating point that is nearly throughput-optimal, an example of which we will demonstrate in this paper in the context of random access. In the rest, we first summarize the main contributions of this paper. Next, we briefly contrast our results with those in recent literature, to highlight the salient points of this work with respect to other related works. This will be followed by the system model, the analysis, numerical examples and conclusions.

There have been previous studies of AoI optimization in scheduled access [13, 16, 17, 18]. MaxWeight type strategies where the transmission probabilities depend on age [1, 8, 19] and CSMA-type policies [20, 21] have also been studied.

Stationary and distributed policies where new packets are generated at will (whereby nodes generate a new sample when they decide to transmit) were considered in [22, 11, 23]. A pioneering study of age in random access [11] bounded the age performance of slotted ALOHA: the time average age achievable by slotted ALOHA in a large network of symmetric nodes is a factor of $2e$ away from an ideal round-robin allocation. In [23], an AoI expression was derived considering up to a certain number of retransmissions of the same packet, in a network using slotted ALOHA.

In Chapter 2, we provide a complete analysis of a recent random access policy, *threshold Aloha*. We describe a detailed comparison between slotted Aloha and threshold Aloha. In Chapter 3, a reservation-based policy is introduced based on threshold Aloha, called *MuMiSTA*. MuMiSTA is thoroughly analyzed in terms of channel throughput and age of information. In Chapter 4, we investigate how the information freshness can be optimized in a wireless energy transfer setting. The optimality of a threshold policy is shown to optimal for the memoryless and Markovian channels.

CHAPTER 2

SLOTTED ALOHA WITH AN AGE THRESHOLD

2.1 System Model

We consider a wireless network containing n sources (alternatively, users) and a common access point (AP). The sources wish to send occasional status updates to their (possibly remote) destinations reached through the AP. Nodes are synchronized with a common time reference (obtained through a control channel), and there is a slotted time-frame structure. We adopt the generate-at-will model [24] such that each source that decides to transmit generates a fresh sample just before transmission (An extension to exogenous arrivals is made in Section 2.2.6). We disallow collision resolution, such that if two or more users attempt transmission in the same slot, all transmitted packets are lost. There are no re-transmissions. When a failed source attempts transmission again, it generates a new packet. If there is no collision, the transmission of the packet is successfully completed within a single time slot.

For simplicity, we will have each source generate a single data flow. The Age of Information (AoI) of user $i \in \{1, \ldots, n\}$ (equivalently, that of flow i) at time slot t, $A_i[t]$, is defined as the number of time slots that have elapsed since the freshest packet of this flow thus far received by the AP was generated. Due to the generate-at-will model we imposed, $A_i[t]$ is equal to the number of slots since the most recent successful transmission of source i, plus one. In the case of a successful transmission, the sender receives a 1-bit acknowledgement (possibly piggybacked on a back-channel packet.), and resets the age of its flow to 1. Accordingly, the age process $\{A_i[t], t = 1, 2, \ldots\}$ evolves as:

$$A_i[t] = \begin{cases} 1, & \text{source } i \text{ transmits successfully at } t-1, \\ A_i[t-1]+1, & \text{otherwise.} \end{cases} \tag{2.1}$$

The long term average AoI of source i is defined as:

$$\Delta_i = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} A_i[t] \tag{2.2}$$

on each sample path where the limit exists. Next, we define the threshold-ALOHA policy.

2.2 Problem Definition and Analysis

In slotted ALOHA, users initiate transmission attempts with a fixed probability τ in each time slot. When buffering and re-transmissions are allowed, this algorithm is unstable. Stabilization can be achieved through modification of the probability τ according to the state of the network, which is often inferred through feedback about successful transmission. In the same vein, feedback about successful transmissions can be used by each source to determine its instantaneous age. In [12], a simple modification of slotted ALOHA was proposed, which we shall refer to as **threshold-ALOHA** in the rest of this thesis. (This algorithm was called Lazy Policy in [12], we modify the name here to one that may be more descriptive of the nature of the policy.)

Threshold-ALOHA is a simple age-aware extension of slotted ALOHA: sources will wait until their age reaches a certain threshold Γ, before they turn on their slotted ALOHA mechanism, and only then start to attempt transmission with a fixed probability τ at each time slot. Hence, sources, who have successfully sent an update not more than $\Gamma - 1$ time slots ago, stay idle and allow others with larger ages contend for the channel. It was numerically observed, without proof, in [12] that this policy is an improvement over slotted ALOHA in the sense that it achieves around half the long term average age achieved by regular slotted ALOHA, without significantly compromising network throughput. Furthermore, it was hypothesized that the optimal threshold scales with the network size as $\Gamma = 2.2n$. These will be confirmed to be essentially correct, as part of the results of our precise analysis of the various convergence modes of this policy.

From the above description of threshold-ALOHA, it is clear that the decision of each source at time slot t is determined by its age at the beginning of this time slot: if the age is below threshold, the node will stay idle, and if not, it will transmit with probability τ. In [12] it was established that the age vector of the sources can be used to denote the state of the network, and for any value of n, this state evolves as a Markov Chain (MC):

$$\mathbf{A}[t] \triangleq \langle A_1[t] \quad A_2[t] \quad \dots \quad A_n[t] \rangle \tag{2.3}$$

It was also shown in [12] that for the purpose of age analysis, it suffices to consider a truncated version of this MC, which constitutes a Finite State Markov Chain (FSMC), with a unique steady-state distribution. The truncated model is based on the observation that once the age of a source exceeds Γ, it becomes an *active* source, and its behavior remains same regardless of how much further its age increases. In most of the remainder of our analysis, unless stated otherwise, the ages of active sources will be truncated at Γ. Due to the ergodicity of the FSMC, and due to the symmetry between the users, the time average AoI (2.2) of each user can be found by computing the expectation over the steady-state distribution of the age, which is equal for all i:

$$\Delta_i = \lim_{t \to \infty} \mathbb{E}\left[A_i[t]\right] \tag{2.4}$$

In the rest, we explore this steady-state distribution and exploit its asymptotic characteristics.

2.2.1 Steady State Solution

As in [12], we define the truncated state vector:

$$\mathbf{A}^{\Gamma}[t] \triangleq \left\langle A_1^{\Gamma}[t] \quad A_2^{\Gamma}[t] \quad \dots \quad A_n^{\Gamma}[t] \right\rangle \tag{2.5}$$

where $A_i^{\Gamma}[t] \in \{1, 2, \dots, \Gamma\}$ is the AoI of source *i* at time $t \in \mathbb{Z}^+$ truncated at Γ and evolves as:

$$A_i^{\Gamma}[t] = \begin{cases} 1, & \text{source } i \text{ updates at time } t-1, \\ \min\left\{A_i^{\Gamma}[t] + 1,\ \Gamma\right\}, & \text{otherwise.} \end{cases} \tag{2.6}$$

The resulting state space is $\mathcal{S} = \{1, 2, \dots, \Gamma\}^n$. As shown in [12], $\{\mathbf{A}^{\Gamma}[t], t \geq 1\}$ is a finite state Markov Chain (MC) with a unique steady state distribution. We first describe the recurrent class.

Proposition 1. *If a state* $\langle s_1 \quad s_2 \quad \ldots \quad s_n\rangle$ *in the truncated MC* $\{\mathbf{A}^\Gamma[t], t \geq 1\}$ *is recurrent, then for distinct indices i and j,* $s_i = s_j$ *if and only if* $s_i = s_j = \Gamma$.

Proof. f Suppose at time $t > 1$, there exist two entries of the state vector that are equal to 1, i.e., there is a pair of sources (i, j) such that $s_i = s_j = 1$. This would imply two simultaneous successful transmissions at $t - 1$. However, this is impossible due to the assumption that colliding packets are lost. We extend this argument to cases where $s_i = s_j = s < \Gamma$ and $t > s$. The existence of such an (i, j) pair implies two simultaneous transmissions at $t - s$. As this is impossible, such (i, j) pairs cannot exist. Finally, if the system started in a state where there are two (or more) users that have the same age, $a < \Gamma$, at $t = 1$, these ages will grow to Γ in $\Gamma - a$ time slots after which they will be decoupled, because only one can get reset to 1 at a time. Therefore, if the initial state of the MC is one that contains non-distinct below-threshold values, the chain will leave this state in at most Γ time slots, and it will never return. This implies that such states are transient. □

According to Prop. 1, states where distinct users have equal below-threshold age are transient. So, without loss of generality, the steady-state analysis that follows will be limited to the remaining states, where $s_i = s_j$ if and only if $s_i = s_j = \Gamma$. It will later be proved that all the remaining states are recurrent, moreover, as there is a unique steady state (from [12]) those states are all in the same recurrent class in the truncated MC. So in the rest, we refer to the remaining states as recurrent states.

We define the *type* of a recurrent state in the following way:

$$T\langle s_1 \quad s_2 \quad \ldots \quad s_n\rangle = (M, \{u_1, u_2, \ldots, u_{n-M}\}), \tag{2.7}$$

where M is the number of entries equal to Γ (*i.e.*, the number of active sources), and the set $\{u_1, u_2, \ldots, u_{n-M}\}$ is the set of entries smaller than Γ (*i.e.*, the set of ages below the threshold).

Proposition 2. *States of the same type have equal steady state probabilities.*

Proof. Follows from the symmetry between users. □

Next, we further show that, for a given M, the set $\{u_1, \ldots, u_{n-M}\}$ has no effect on the steady state probability of a state. In other words, this probability is determined by M, the number of active sources. This facilitates the derivation of the distribution of the number of active sources.

Lemma 1. *The truncated MC $\{\mathbf{A}^\Gamma[t], t \geq 1\}$ has the following properties:*

(i) Given a state vector $\langle s_1 \quad s_2 \quad \ldots \quad s_n \rangle$, its steady state probability depends only on the number of entries that are equal to Γ.

(ii) Let P_m be the total steady state probability of states having m active users. Then
$$\frac{P_m}{P_{m-1}} = \frac{(1-(m-1)\tau(1-\tau)^{m-2})(n-m+1)}{m\tau(1-\tau)^{m-1}(\Gamma-1-n+m)}$$
.

(iii) P_m is explicitly given as (2.17) for $m \geq 0$.

Proof. First, suppose that the given state vector has no entry equal to 1. Let the type of this state vector be $\mathcal{T}_1 \triangleq (M, \{u_1, u_2, \ldots, u_{n-M}\})$, where $M \in \{0, 1, \ldots, n\}$ is the number of entries equal to Γ and $u_i > 1, i = 1, 2, \ldots, u - M$. As there is no source whose age is 1 at the current time, t, there has been no successful transmission in the previous time slot, $t-1$. Hence, the number of active users at $t-1$ cannot have been $M+1$ or larger. So the state at $t-1$ must be one of the following types:

- $\mathcal{T}_2 \triangleq (M, \{u_1 - 1, u_2 - 1, \ldots, u_{n-M} - 1\})$
- $\mathcal{T}_3 \triangleq (M-1, \{\Gamma - 1, u_1 - 1, u_2 - 1, \ldots, u_{n-M} - 1\})$

If, on the other hand, there was a successful transmission whilst in types $\mathcal{T}_2$ and $\mathcal{T}_3$, the resulting state would have been of type $\mathcal{T}_0 \triangleq (M-1, \{u_1, u_2, \ldots, u_{n-M}, 1\})$.

Alternatively, if the given state vector has an entry that is equal to 1 at current time, t, it indicates a successful transmission at $t-1$. In this case, the given state vector is of type $\mathcal{T}_0$ and the state at $t-1$ must be of types $\mathcal{T}_2$ or $\mathcal{T}_3$, as defined above.

Let $\mathcal{C}_t$ be the set of states that are of type $\mathcal{T}_0$ or type $\mathcal{T}_1$. Let $\mathcal{C}_{t-1}$ be the set of states that are of type $\mathcal{T}_2$ or type $\mathcal{T}_3$. If the system is in a state that is in $\mathcal{C}_t$ at time t, then its state at time $(t-1)$ must be in $\mathcal{C}_{t-1}$. This follows from the fact that there can be at most 1 transmission at each time slot and due to Prop. 1 all source states except Γ are unique. Similarly, if the system is in a state that is in $\mathcal{C}_{t-1}$ at time $(t-1)$, then its state at time t must be in $\mathcal{C}_t$.

Any given state of type $\mathcal{T}_2$ evolves into a state of type $\mathcal{T}_0$ with probability $M\tau(1-\tau)^{M-1}$ and into a state of type $\mathcal{T}_1$ with probability $1-M\tau(1-\tau)^{M-1}$. A state of type $\mathcal{T}_3$ evolves into a state of type $\mathcal{T}_0$ with probability $(M-1)\tau(1-\tau)^{M-2}$ and into a state of type $\mathcal{T}_1$ with probability $1-(M-1)\tau(1-\tau)^{M-2}$. Let $\pi_{\mathcal{T}_j}$ be the steady state probability of a single state of type $\mathcal{T}_j$. By the arguments above, the steady-state probabilities are related to each other by the following equations:

$$\pi_{\mathcal{T}_1} = \pi_{\mathcal{T}_2}(1-M\tau(1-\tau)^{M-1}) + \pi_{\mathcal{T}_3}M(1-(M-1)\tau(1-\tau)^{M-2}) \tag{2.8}$$

$$\pi_{\mathcal{T}_0} = \pi_{\mathcal{T}_2}\tau(1-\tau)^{M-1} + \pi_{\mathcal{T}_3}(M-1)\tau(1-\tau)^{M-2} \tag{2.9}$$

As $\mathbf{A}^\Gamma$ has a unique steady state, a solution set satisfying the above steady state equations shall yield the steady state probabilities. As (2.8) and (2.9) stand for all the incoming and outgoing transition probabilities of all recurrent states, this set of equations fully describes the steady state probabilities. Part (*i*) of our claim can be tested by assigning π_m as the steady state probabilities of system states that have m sources at state Γ. Noting that $\pi_{\mathcal{T}_1} = \pi_{\mathcal{T}_2} = \pi_M$ and $\pi_{\mathcal{T}_0} = \pi_{\mathcal{T}_3} = \pi_{M-1}$, with appropriate substitutions (2.8) becomes:

$$\pi_M = \pi_M(1-M\tau(1-\tau)^{M-1}) + \pi_{M-1}M(1-(M-1)\tau(1-\tau)^{M-2}), \tag{2.10}$$

and (2.9) becomes:

$$\pi_{M-1} = \pi_M\tau(1-\tau)^{M-1} + \pi_{M-1}(M-1)\tau(1-\tau)^{M-2}. \tag{2.11}$$

Both of these equations are reduced to the same equation below that holds for all m:

$$\frac{\pi_m}{\pi_{m-1}} = \frac{1-(m-1)\tau(1-\tau)^{m-2}}{\tau(1-\tau)^{m-1}}. \tag{2.12}$$

Therefore, part (*i*) holds and this can be used to calculate the steady state probability of having m active users. The total number of states corresponding to π_m are the number of recurrent system states with m sources at truncated age Γ:

$$N_m = \binom{n}{m}\frac{(\Gamma-1)!}{(\Gamma-n-1+m)!} \tag{2.13}$$

Recall that P_m was defined as the total probability of all states with m active sources. By Lemma 1 (i), each of these states are equiprobable with steady state probability π_m. Hence,

$$P_m = N_m \pi_m, \tag{2.14}$$

$$\frac{P_m}{P_{m-1}} = \frac{(1-(m-1)\tau(1-\tau)^{m-2})(n-m+1)}{\tau(1-\tau)^{m-1}m(\Gamma-1-n+m)}. \tag{2.15}$$

From (2.15),

$$P_0 = \frac{1}{1+\sum_{m=1}^{n}\prod_{i=1}^{m}\frac{(1-(i-1)\tau(1-\tau)^{i-2})(n-i+1)}{i\tau(1-\tau)^{i-1}(\Gamma-1-n+i)}}, \tag{2.16}$$

$$P_m = P_0 \prod_{i=1}^{m} \frac{(1-(i-1)\tau(1-\tau)^{i-2})(n-i+1)}{\tau(1-\tau)^{i-1}i(\Gamma-1-n+i)}. \tag{2.17}$$

provides the steady state solution. $\square$

2.2.2 Pivoted Markov Chain

In this part, we make our analysis over a single source, which we refer to as the *pivot* source. Any source in the network can be selected as pivot. After selecting a source as pivot, we modify the truncated MC of previous subsection, $\{\mathbf{A}^{\Gamma}[t], t \geq 1\}$, to create *pivoted MC* $\{\mathbf{P}^{\Gamma}[t], t \geq 1\}$, where the states of all the sources except the pivot are truncated at Γ.

We extend our definitions and arguments from the proof of Lemma 1 to $\mathbf{P}^{\Gamma}$, in particular extend the definition of types of states. The *type* of a state in $\mathbf{P}^{\Gamma}$ is defined as:

$$\mathrm{T}^{\mathbf{P}}\langle S^{\mathbf{P}}\rangle \triangleq (s, M, \{u_1, u_2, \ldots, u_{n-M-1}\}), \tag{2.18}$$

where $s \in \mathbb{Z}^+$ is the state of the pivot source, M is the number of entries equal to Γ (*i.e.*, the number of active sources not including the pivot), and the set is the set of entries smaller than Γ (*i.e.*, the set of ages below the threshold, not including s). With a slight abuse of notation, we will refer to such a state as *type M-state* where it is clear from the context.

Proposition 3. *(i) $\boldsymbol{P}^{\Gamma}$ has a unique steady state distribution.*

(ii) Steady state probability of a type-m state in $\boldsymbol{P}^{\Gamma}$ is equal to π_m, obeying (2.12), if $s \in \{1, 2, \ldots, \Gamma - 1\}$.

Proof. States in $\mathbf{P}^{\Gamma}$ where $s = 1, 2, \ldots, \Gamma - 1$ have one-to-one correspondence with the related states in the truncated MC $\mathbf{A}^{\Gamma}$. The system visiting these corresponding states in $\mathbf{P}^{\Gamma}$ and $\mathbf{A}^{\Gamma}$ constitutes the same event hence these have identical steady state probabilities and identical transition probabilities, by construction. Therefore, they follow (2.12).

Next, we shall establish the existence of a steady state probability for the states in $\mathbf{P}^{\Gamma}$ for which $s \geq \Gamma$. For a given s, we augment $\mathbf{A}^{\Gamma}$ to form the *augmented truncated MC* $\{\mathbf{A}^{s,\Gamma}[t], t \geq 1\}$ where the pivot is truncated at $s+1$ and all other sources are truncated at Γ. Truncation of the pivot source is illustrated in Fig. 2.1. Let us the call the state where the state of the pivot source is $s + 1$ and state of all other sources is Γ the *unlucky* state. The unlucky state can be reached by all the states in the MC, including the unlucky state itself, if there are no successful transmissions in the network for s consecutive time slots, which can happen with non-zero probability. This means that there is a single recurrent class in this MC and a unique steady state distribution. Finally, there is a one-to-one correspondence between the states of $\mathbf{A}^{s,\Gamma}$ and $\mathbf{P}^{\Gamma}$ for which the state of the pivot source is s. Existence of steady state probabilities for the states in $\mathbf{A}^{s,\Gamma}$ entails the existence of steady state probabilities for the states in $\mathbf{P}^{\Gamma}$. $\square$

Definition 1. *Let $S^{\boldsymbol{P}}$ be a state in $\boldsymbol{P}^{\Gamma}$ of type $T^{\boldsymbol{P}}\langle S^{\boldsymbol{P}}\rangle = (s, m, \{u_1, u_2, \ldots, u_{n-m-1}\})$, where the $\{u_i\}$ are ordered from largest to smallest. $Q(S^{\boldsymbol{P}})$, preceding type of $S^{\boldsymbol{P}}$, is defined as follows:*

$$Q(S^{\boldsymbol{P}}) = \begin{cases} T^{\boldsymbol{P}}\langle S^{\boldsymbol{P}}\rangle, & \text{if} \quad s = 1 \\ (s-1, m, \{\Gamma - 1, u_1 - 1, \\ u_2 - 1, \ldots, u_{n-m-2} - 1\}), & \text{if} \quad s \neq 1,\ u_{n-m-1} = 1 \\ (s-1, m, \{u_1 - 1, u_2 - 1, \\ \ldots, u_{n-m-1} - 1\}), & \text{if} \quad s \neq 1,\ u_{n-m-1} \neq 1 \end{cases} \tag{2.19}$$

The reasoning behind $Q(S^{\mathbf{P}})$ is that if current state is $S^{\mathbf{P}}$ and number of active sources did not change in the previous time slot (excluding pivot source), then the type of previous state must be $Q(S^{\mathbf{P}})$. This does not hold for case $s = 1$, but we are not interested

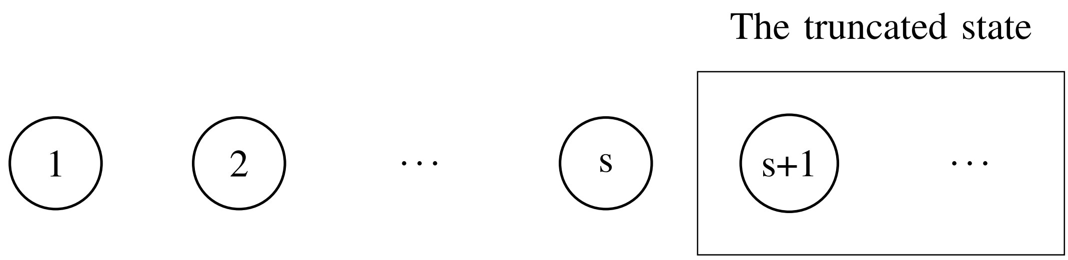


Figure 2.1: States of the pivot source in $\mathbf{A}^{s,\Gamma}$ compared to $\mathbf{P}^{\Gamma}$.

in such a characterization for this case; nevertheless, we choose $Q(S^{\mathbf{P}})$ to be the type $S^{\mathbf{P}}$ itself, so that we do not have to exclude this special case in what follows. Finally, we denote the steady state probability of $S^{\mathbf{P}}$ as $\pi(S^{\mathbf{P}})$ or $\pi(s, m, \{u_1, u_2, \ldots, u_{n-m-1}\})$.

Lemma 2. *Let $S_1^{\boldsymbol{P}}$ and $S_2^{\boldsymbol{P}}$ be two arbitrary states in $\boldsymbol{P}^{\Gamma}$ where the state of the pivot source is equal for both states. Let the types of $S_1^{\boldsymbol{P}}$ and $S_2^{\boldsymbol{P}}$ be:*

$$\boldsymbol{T}^{\boldsymbol{P}}\langle S_1^{\boldsymbol{P}}\rangle = (s, m_1, \{u_1, u_2, \ldots, u_{n-m_1-1}\})$$

$$\boldsymbol{T}^{\boldsymbol{P}}\langle S_2^{\boldsymbol{P}}\rangle = (s, m_2, \{v_1, v_2, \ldots, v_{n-m_2-1}\})$$

i) Let $Q_1^{\boldsymbol{P}}$ be any state satisfying $\boldsymbol{T}^{\boldsymbol{P}}\langle Q_1^{\boldsymbol{P}}\rangle = Q(S_1^{\boldsymbol{P}})$. Then,

$$\lim_{n\to\infty} \frac{\pi(S_1^{\boldsymbol{P}})}{\pi(Q_1^{\boldsymbol{P}})} = 1 \tag{2.20}$$

ii) If $m_1 = m_2$, then

$$\lim_{n\to\infty} \frac{\pi(S_1^{\boldsymbol{P}})}{\pi(S_2^{\boldsymbol{P}})} = 1 \tag{2.21}$$

iii) If $m_1 = m_2 + 1$, then

$$\lim_{n\to\infty} \frac{\pi(S_1^{\boldsymbol{P}})}{n\,\pi(S_2^{\boldsymbol{P}})} = \frac{e^{k\alpha}}{\alpha} - k, \tag{2.22}$$

where $\lim_{n\to\infty} \frac{m_1}{n} = k$ and $\lim_{n\to\infty} \tau n = \alpha$. ($k, \alpha \in \mathbb{R}^+$)

Proof. See Appendix A. □

Theorem 1. *For some $r, \alpha \in \mathbb{R}^+$, such that $\lim_{n\to\infty} \frac{\Gamma}{n} = r$ and $\lim_{n\to\infty} \tau n = \alpha$, define $f : (0, 1) \to \mathbb{R}$:*

$$f(x) = \ln(\frac{e^{x\alpha}}{x\alpha} - 1) + \ln(\frac{r}{x + r - 1} - 1). \tag{2.23}$$

Then, for all m *such that* $\lim_{n\to\infty} \frac{m}{n} = k \in (0,1)$ *and* $s \in \mathbb{Z}^+$

$$\lim_{n\to\infty} \ln \frac{P_m^{(s)}}{P_{m-1}^{(s)}} = f(k), \tag{2.24}$$

where $P_m^{(s)}$ *is the steady state probability of having* m *active sources (excluding the pivot source), given that state of the pivot source is* s.

Proof. The term $P_m^{(s)}$ is the total steady state probability of states in which there are m active users and the state of the pivot source is s. The number of such recurrent states is:

$$N_m = \binom{n-1}{m} \frac{(\Gamma-1)!}{(\Gamma-n+m)!}. \tag{2.25}$$

Meanwhile, the number of recurrent states containing $m-1$ active users is:

$$N_{m-1} = \binom{n-1}{m-1} \frac{(\Gamma-1)!}{(\Gamma-n+m-1)!}. \tag{2.26}$$

Let $\mathcal{B}_m = \{S_1^{(m)}, S_2^{(m)}, \ldots, S_{N_m}^{(m)}\}$ be the set of all recurrent type-m states where the state of the pivot source is s.

Similarly, we define the set $\mathcal{B}_{m-1} = \{S_1^{(m-1)}, S_2^{(m-1)}, \ldots, S_{N_{m-1}}^{(m-1)}\}$ as the set of all recurrent type-$(m-1)$ states where the state of the pivot source is s. Then,

$$\begin{aligned}
\lim_{n\to\infty} \frac{P_m^{(s)}}{P_{m-1}^{(s)}} &= \lim_{n\to\infty} \frac{\sum_{i=1}^{N_m} \pi(S_i^{(m)})}{\sum_{j=1}^{N_{m-1}} \pi(S_j^{(m-1)})} \qquad (2.27) \\
&\overset{(a)}{=} \lim_{n\to\infty} \frac{n\sum_{i=1}^{N_m} \left[\pi(S_i^{(m)})/n\pi(S_1^{(m-1)})\right]}{\sum_{j=1}^{N_{m-1}} \left[\pi(S_j^{(m-1)})/\pi(S_1^{(m-1)})\right]} \\
&\overset{(b)}{=} \lim_{n\to\infty} \frac{n\sum_{i=1}^{N_m} (\frac{e^{k\alpha}}{\alpha} - k)}{\sum_{j=1}^{N_{m-1}} 1} \\
&= \lim_{n\to\infty} \frac{nN_m(\frac{e^{k\alpha}}{\alpha} - k)}{N_{m-1}} \\
&= \lim_{n\to\infty} \frac{n(n-m)(\frac{e^{k\alpha}}{\alpha} - k)}{m(\Gamma-n+m)}
\end{aligned}$$

$$= \left(\frac{e^{k\alpha}}{k\alpha} - 1\right)\left(\frac{1-k}{r+k-1}\right),$$

where (a) is obtained by by dividing both sides of the fraction by the steady state probability of any element of $\mathcal{B}_{m-1}$, which was arbitrarily chosen as the first element, and (b) follows from Lemma 2 (ii) and (iii). Hence,

$$\lim_{n\to\infty} \ln \frac{P_m^{(s)}}{P_{m-1}^{(s)}} = \ln(\frac{e^{k\alpha}}{k\alpha} - 1) + \ln(\frac{r}{r+k-1} - 1) = f(k). \tag{2.28}$$

$\square$

The above argument shows that as $n \to \infty$ the relation $P_m^{(s)}/P_{m-1}^{(s)}$ determines the PMF of m regardless of the state s of the pivot source. Consequently, the number of active sources (excluding the pivot), m, is independent of the state of the pivot source. We record this in the following corollary:

Corollary 1. *In the case of a large network ($n \to \infty$),*

(i) The number of active sources, m, (excluding the pivot) is independent of the state s of the pivot source.

(ii) As long as $s \geq \Gamma$, the probability of a successful transmission being made by the pivot source is $\tau(1-\tau)^m$ which has no dependence on s.

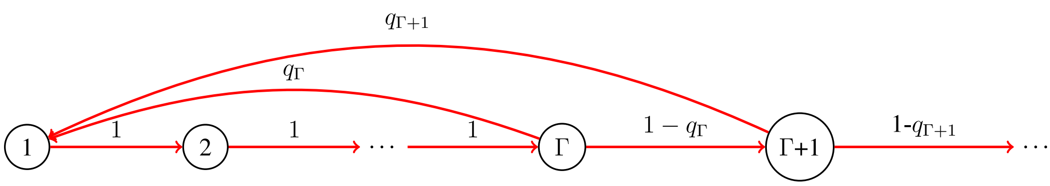
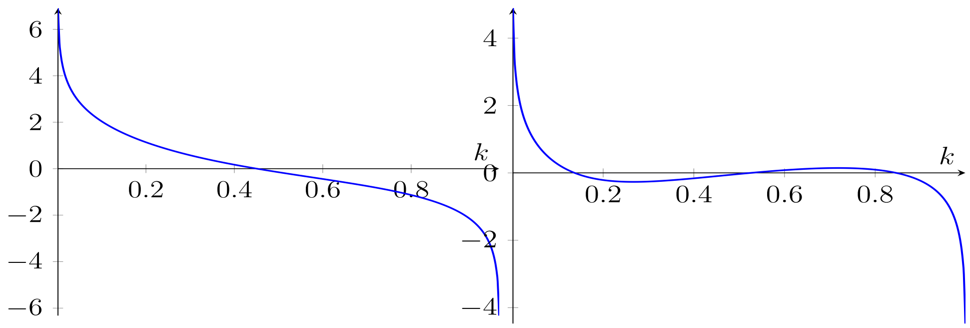


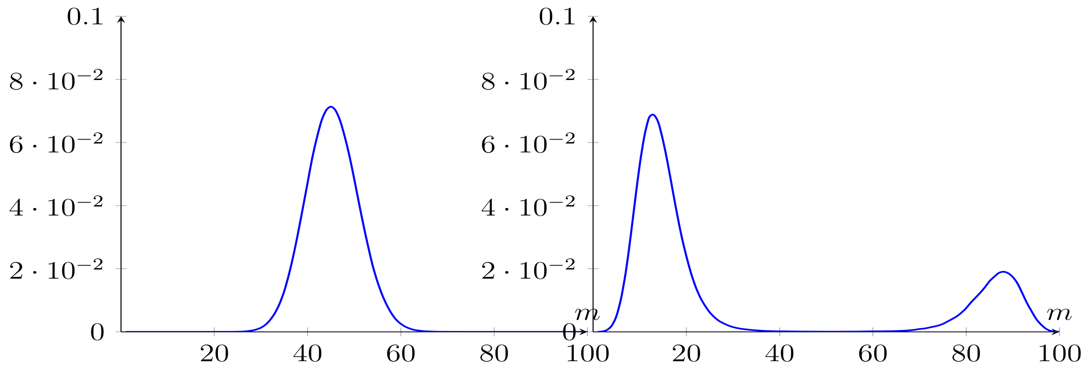
Figure 2.2: State diagram of the pivot source

The transition probabilities q_s marked on Fig. 2.2 refer to the probability of a successful transmission made by the pivot source. In the rest, we will consider the asymptotic case as the network size n grows. We will show that in the limit as $n \to \infty$, q_s is equal to some q_o for all values of s as long as the pivot source is active.



(a) Single root case ($\alpha = 2$, $r = 1.5$) (b) Three-root case ($\alpha = 5$, $r = 2.5$)

Figure 2.3: Plot of $f(k)$



(a) Single root case ($\alpha = 2$, $r = 1.5$) (b) Three-root case ($\alpha = 5$, $r = 2.5$)

Figure 2.4: PMF of m $(n = 100)$

2.2.3 Large network asymptotics

In this part, we investigate the PMF of m, number of active sources in the network. Function f of Theorem 1 gives valuable insight on the distribution of m and we will derive some properties of f with the eventual goal of proving that the ratio of active users, k, converges to the root of f in probability, presented in Theorem 2.

To facilitate the asymptotic analysis in the network size n, we replace the main parameters of the model, τ and Γ, with the following that control the scaling of these parameters with n. As the number of active sources, m, takes values between 0 and

n, the fraction of active sources, k, will vary between 0 and 1.

$$\alpha = n\tau, \quad r = \Gamma/n, \quad k = m/n \tag{2.29}$$

Proposition 4. *Roots of f for which f is decreasing correspond one-to-one to the local maxima of P_m, with a scale of n.*

In this context, α and r are fixed system parameters while k, the fraction of active users, is a variable indicating the instantaneous system load. As the change in P_m is determined by $f(k)$, the roots of $f(k)$ provide the local extrema of P_m. Local maxima of P_m are the points where both $\ln P_m/P_{m-1}$ and $\ln P_m/P_{m+1}$ are positive, corresponding to roots of $f(k)$ for which f is decreasing. This correlation is visible in Figures 2.3 and 2.4. The following proposition restricts the number of roots $f(k)$, and therefore the number of local maxima P_m can have.

Proposition 5. *The number of distinct roots of f is at least 1 and at most 3.*

Proof. To prove that f has at least 1 root, it is sufficient to observe that $f(0^+) = +\infty$ and $f(1^-) = -\infty$. Since f is continuous in (0,1) domain, f has at least one root.

To prove that f has at most 3 roots, we formulate r in terms of α and k when $f(k) = 0$.

$$f(k) = \ln(\frac{e^{k\alpha}}{k\alpha} - 1) + \ln(\frac{r}{k+r-1} - 1) = 0 \tag{2.30}$$

$$r = \frac{e^{k\alpha}(1-k)}{k\alpha} \tag{2.31}$$

$$\frac{dr}{dk} = \frac{e^{k\alpha}}{k^2\alpha}(-\alpha k^2 + \alpha k - 1) \tag{2.32}$$

Since $\frac{dr}{dk}$ has at most two roots, there can be at most 3 different values of k that satisfy (2.31). These are the only possible roots of $f(k)$. Hence, $f(k)$ has at most 3 roots. $\square$

Since $f(k)$ has at most three roots, there can be at most 2 roots of f where f is decreasing and consequently at most two local maxima. Cases of one local maximum and two local maxima are analyzed separately, however they lead to a similar discussion. Theorem 2 is given for the case where $f(k)$ has only one root and a single local maximum. The case with 2 local maxima is discussed in Section 2.2.4.

Theorem 2. *Let k_0 be the only root of $f(k)$ and m be the number of active sources. For the sequence $\epsilon_n = cn^{-1/3}$ where $c \in \mathbb{R}^+$,*

$$\Pr\left(\left|\frac{m}{n} - k_0\right| < \epsilon_n\right) \to 1 \tag{2.33}$$

Proof. See Appendix B. □

This theorem establishes that the fraction of active users converges in probability to k_0 as the network size grows. Loosely speaking, threshold-ALOHA gradually converts the system to one with nk_0 users with a slotted ALOHA analysis. At steady state, approximately nk_0 sources will be making transmission attempts while remaining $n - nk_0$ sources with small age will be idle. For this reason, it resembles a stabilized ALOHA algorithm. For large n, throughput of the channel remains close to e^{-1} while average age can be dramatically improved through optimal parameters, as will be shown in the Section 2.2.5.

2.2.4 Double Peak Case

In this section, we extend the single peak analysis of the previous section to the case with 2 peaks. Theorem 3 gives the same result as in Theorem 2, although it imposes an additional integral constraint to be applicable.

So far, it has been argued that roots of $f(k)$ where f is decreasing correspond to the peaks in the probability distribution of the number of active sources. If there are two such roots, then there will be two possible values of m where the number of active sources are concentrated around. Accordingly, we define the following state sets:

$$\mathcal{S}_0 \triangleq \left\{ S \mid T\left<S\right> = (m, \{\ldots\}) \text{ w. } \frac{m}{n} \leq \frac{k_0 + k_1}{2} \right\} \tag{2.34}$$

$$\mathcal{S}_1 \triangleq \left\{ S \mid T\left<S\right> = (m, \{\ldots\}) \text{ w. } \frac{k_0 + k_1}{2} < \frac{m}{n} < \frac{k_1 + k_2}{2} \right\} \tag{2.35}$$

$$\mathcal{S}_2 \triangleq \left\{ S \mid T\left<S\right> = (m, \{\ldots\}) \text{ w. } \frac{k_1 + k_2}{2} \leq \frac{m}{n} \right\} \tag{2.36}$$

$\mathcal{S}_0$ corresponds to the states where number of active users are around the smaller root and $\mathcal{S}_2$ corresponds to the states where number of active users are around the larger

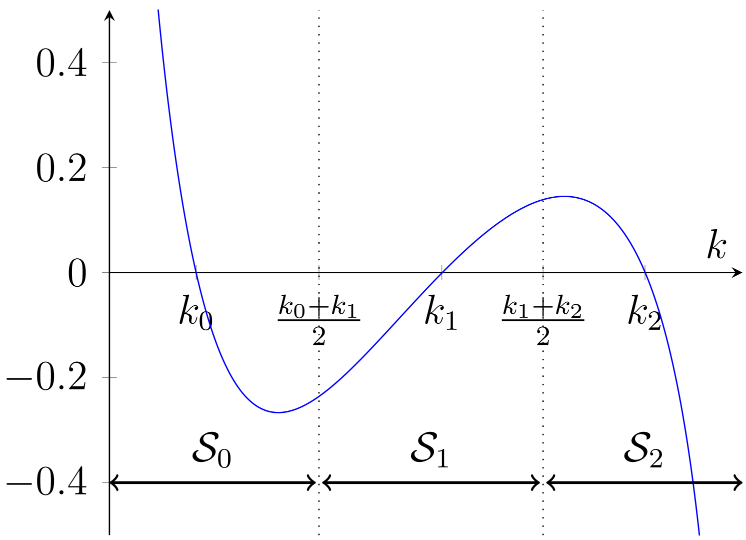


Figure 2.5: State sets

root. States in between are grouped as $\mathcal{S}_1$ and thresholds are set at the mid-points between consecutive roots.

In the proof of Theorem 3, it is shown that, if the integral is negative, probability of $\mathcal{S}_1$ and $\mathcal{S}_2$ state sets diminishes as n goes to infinity. By showing that $\mathcal{S}_0$ happens with probability 1, basic principles used for the single peak case can be used again to derive similar results.

Theorem 3. *Let f(k) have three distinct roots and k_0, k_1, k_2 be the roots in increasing order and m be the number of active sources.*

i) If

$$\int_{k_0}^{k_2} f(k)dk < 0 \tag{2.37}$$

then for the sequence $\epsilon_n = cn^{-1/3}$ where $c \in \mathbb{R}^+$,

$$\Pr\left(\left|\frac{m}{n} - k_0\right| < \epsilon_n\right) \to 1 \tag{2.38}$$

ii) If

$$\int_{k_0}^{k_2} f(k)dk > 0 \tag{2.39}$$

then for the sequence $\epsilon_n = cn^{-1/3}$ *where* $c \in \mathbb{R}^+$*,*

$$\Pr\left(\left|\frac{m}{n} - k_2\right| < \epsilon_n\right) \to 1 \tag{2.40}$$

Proof. See Appendix C. $\square$

The ratio of active users converges to either k_0 or k_2, depending on the sign of the integral above. If the integral result is positive, this ratio will converge to the larger root, however, this is not desired since larger root is equivalent to more active users at the same time. In order to fully benefit from the age threshold, parameters should be chosen such that k converges to k_0.

Even though Theorem 3 yields a similar result as in Theorem 2, double peak cases may not be as practical as single peak cases in networks with fewer users. For n values that are not large enough, steady state probabilities of $\mathcal{S}_1$ and $\mathcal{S}_2$ may not be small enough to yield useful results. As k values for state sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are larger than that for $\mathcal{S}_0$, these states have more active users, which may lead to the congestion of the channel by having too many users trying to transmit at the same time. This negates the benefit of threshold-ALOHA and should be avoided. Single peak cases do not have $\mathcal{S}_1$ and $\mathcal{S}_2$ sets and system converges more quickly to k_0.

In networks with a large number of users, initial conditions must be selected properly to achieve good results. Selecting all users active initially leads to the aforementioned congestion scenarios, slowing down the convergence in Theorem 3. As n increases, the transition probabilities between state sets decrease exponentially. If the initial state of the system is in $\mathcal{S}_2$, it may be nearly impossible for the network to reach a state in $\mathcal{S}_0$ in a reasonable time period. Initial state of users can be randomized to prevent initial congestion. Despite all these drawbacks, the double peak cases produce asymptotically optimal values and are preferable as network size increases.

2.2.5 Steady state average AoI in the large network limit

Theorem 4. *Optimal parameters for threshold-ALOHA in an infinitely large network satisfy the following:*

$$\lim_{n\to\infty} \frac{\Gamma^*}{n} = 2.21 \tag{2.41}$$

$$\lim_{n\to\infty} n\tau^* = 4.69 \tag{2.42}$$

Moreover, the optimal expected AoI at steady state scales as:

$$\lim_{n\to\infty} \frac{\Delta^*}{n} = 1.4169 \tag{2.43}$$

Proof. As can be recalled from the ending of Section 2.2.2, q_0 was defined as successful transmission probability of an active source and it has been argued that q_0 is independent of the age of the active source. Alternatively, q_0 can be expressed as:

$$q_0 = \mathbb{E}[\tau(1-\tau)^{M-1}], \tag{2.44}$$

where the expectation is over the distribution of M, the number of active sources at steady state, which was characterized earlier. We firstly prove that

$$\lim_{n\to\infty} n\, q_0 = \alpha e^{-k_0\alpha}. \tag{2.45}$$

Let γ_n be defined as:

$$\gamma_n \triangleq \Pr(m_0 - cn^{2/3} < M < m_0 + cn^{2/3}), \tag{2.46}$$

where $m_0 = k_0 n$. From Theorem 2 and 3, $\gamma_n \to 1$ as $n \to \infty$. When M is within the bounds given in (2.46), the successful transmission probability is also bounded from both sides. This is used to obtain the following bound:

$$\begin{split}\gamma_n[\tau(1-\tau)^{m_0}(1-\tau)^{-cn^{2/3}}] < q_0 < (1-\gamma_n) + \\ \gamma_n[\tau(1-\tau)^{m_0}(1-\tau)^{cn^{2/3}}]\end{split} \tag{2.47}$$

As n goes to infinity, both upper and lower bounds converge to $\tau(1-\tau)^{m_0}$. Finally,

$$\lim_{n\to\infty} n\, q_0 = \lim_{n\to\infty} n\tau(1-\tau)^{m_0} = \alpha e^{-k_0\alpha} \tag{2.48}$$

	r^*	α^*	k_0^*	G	Δ^*/n	$Thr.$
TA (SP)	2.17	4.43	0.2052	0.9090	1.4226	0.3658
TA (DP)	2.21	4.69	0.1915	0.8981	**1.4169**	0.3644
SA	0	1	1	1	e	e^{-1}

Table 2.1: A comparison of optimized parameters of ordinary slotted ALOHA and threshold-ALOHA, and the resulting AoI and throughput values. TA: Threshold-ALOHA, SP: Single-Peak, DP: Double Peak, SA: Slotted ALOHA, r^*: Γ/n; α^*: transmission probability$\times n$; k_0^*: expected fraction of active users; G: expected number of transmission attempts per slot; Δ^*: avg. AoI, Thr: Throughput

Value of q_0 can be used to compute steady state probabilities of a single source using the model in Fig. 2.2. In this model, states are not truncated and age is equivalent to state. Steady state probability of state j is:

$$\pi_j = \frac{(1-q_0)^{\max\{j-\Gamma,0\}}}{\Gamma - 1 + 1/q_0}, \quad j = 1, 2, \ldots \tag{2.49}$$

Steady state probabilities are used to derive the following expected time-average AoI expression:

$$\Delta = \frac{\Gamma(\Gamma-1)}{2(\Gamma - 1 + 1/q_0)} + 1/q_0 \tag{2.50}$$

Limiting behavior of average AoI is found as:

$$\lim_{n\to\infty} \frac{\Delta}{n} = \frac{r^2}{2(r + e^{k_0\alpha}/\alpha)} + e^{k_0\alpha}/\alpha \tag{2.51}$$

(2.51) can alternatively be expressed in terms of r and k_0:

$$\lim_{n\to\infty} \frac{\Delta}{n} = r\frac{k_0^2+1}{2(1-k_0)} \tag{2.52}$$

Average AoI can be optimized by searching values of r and α that minimizes (2.51). □

Optimal parameters and steady-state characteristics (expected fraction of active users, expected average AoI and throughput) of threshold-ALOHA derived from (2.51) are summarized in Table 2.1 and contrasted with those of regular slotted ALOHA as a reference. Note that as threshold-ALOHA has two possible operating regimes, results for these, namely the single peak case and double peak case are separately provided. Note that slotted ALOHA is a special case of threshold-ALOHA where the age threshold is $\Gamma = 1$ and all users are active regardless of their ages, and thus $r = 1/n$ goes to 0, from (2.29).

In Table 2.1, G refers to the expected number of transmission attempts in a single slot. Under threshold-ALOHA, G is equal to the the product of τ, probability of a transmission attempt, and nk_0, number of active users. As a result, $G = k_0\alpha$ holds. Value of G can be used to compare the throughput of basic slotted ALOHA and threshold-ALOHA. Ge^{-G} is the probability of a successful transmission under both of these policies, since

$$\lim_{n\to\infty} nk_0q_0 = k_0\alpha e^{-k_0\alpha} = Ge^{-G}. \tag{2.53}$$

Hence, the probability of a successful transmission is upper bounded by e^{-1}, with equality if $G = 1$. Under an AoI-optimized selection of Γ and τ for TA, G is equal to 0.8981, for which the throughput is 0.3644. Note that the throughput drop from the upperbound is below 1 percent, in return for reduction in AoI to almost half of what is achievable with slotted ALOHA.

The AoI in slotted ALOHA under optimal parameters is [11]:

$$\Delta = \frac{1}{2} + \frac{1}{\tau(1-\tau)^{n-1}}. \tag{2.54}$$

The expression in (2.54) can be minimized by setting $\tau = 1/n$. Hence, optimal AoI under slotted ALOHA has the following limit [23]:

$$\lim_{n\to\infty} \frac{\Delta^{SA}}{n} = \lim_{n\to\infty} \frac{1}{2n} + \frac{1}{\left(1-\frac{1}{n}\right)^{n-1}} = e. \tag{2.55}$$

Finally, we observe a similarity between threshold-ALOHA and Rivest's stabilized slotted ALOHA [25, Sec. 4.2.3]. Rivest's algorithm uses collision feedback to estimate the number of active sources, $\hat{m}(t)$, in each time slot and uses this estimate to optimize the probability of transmission, $\tau(t)$, such that $\hat{m}\tau = 1$. Rivest's algorithm has also been exploited in [1] to achieve *age-based thinning*. Even though threshold-ALOHA does not track the number of active users, we have showed that the number of active users converges in probability to some $m_0 = nk_0$ (from (2.33)), and that under optimized parameter settings, $m_0\tau$ is close to 1, similarly to what Rivest's stabilized ALOHA tries to achieve.

2.2.6 Extension to Exogenous Arrivals

The analysis so far has been concerned with a model where sources generate new packets *at will* when they decide to transmit. We will now discuss how our analysis may be extended to a model involving exogenous packet arrival process: At each time slot, a new packet arrives at source i with probability λ_i, independently over users and time slots. Arrivals occur frequently enough such that $\lim_{n\to\infty} n\lambda_i = \infty$. If a packet arrival happens at time slot t, then $a_i(t) = 1$ and $a_i(t) = 0$ otherwise. If, upon an arrival, the source already has a packet that has not been successfully transmitted, the older packet is discarded and replaced by the new one.

In order to provide a lower bound on the performance of TA under these conditions, we relax the policy to one where sources are permitted to make a transmission attempt after Γ time slots even if they have not generated a new packet since their last successful transmission. If no new packet has been generated, the packet available at the source is identical to the most recent packet that was sent to the destination and another successful transmission of this packet would not improve the age. However, this assumption is useful for the extension of our findings onto this case and its analysis provides an upper bound on the optimal age due to its inferiority.

Note that transmission decisions are independent of the arrival times. As packet arrival times do not influence when sources will make a transmission attempts and vice versa, packet generation times and delivery times are independent of each other.

We define the age of flow i at the source as $A_i^s[t]$ and the age of flow i at the destination as $A_i[t]$. The ages refer to time between the current time (synchronized throughout the network) and the creation time of the most recent packet available at the respective location. As such, $A_i^s[t]$ and $A_i[t]$ evolve as:

$$A_i^s[t] = \begin{cases} A_i^s[t-1] + 1, & a_i(t) = 0 \\ 0, & a_i(t) = 1 \end{cases} \tag{2.56}$$

and

$$A_i[t] = \begin{cases} A_i^s[t-1]+1, & \text{src. } i \text{ update at time } t-1 \\ A_i[t-1]+1, & \text{otherwise} \end{cases} \tag{2.57}$$

We define $U_k^{(i)}$ to be the time of k^{th} successful transmission made by source i. Finally, $T_i[t]$ is defined as the time elapsed since the last successful transmission was made by source i, corresponding to the the age process of our original model.

$$T_i[t] = t - \max\{U_k^{(i)} : U_k^{(i)} < t\} \tag{2.58}$$

As a result, $A_i[t]$ can also be formulated as:

$$\begin{aligned} A_i[t] &= A_i^s[t - T_i[t]] + T_i[t] \\ &= A_i^s\left[\max\{U_k^{(i)} : U_k^{(i)} < t\}\right] + T_i[t]. \end{aligned} \tag{2.59}$$

We refer to the average of $T_i[t]$ as Δ_i^{TA}, which was formulated as the average age of the original model in (2.51).

$$\Delta_i^{\text{TA}} = \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} T_i[t] \tag{2.60}$$

Let $I_i[k]$ be the time between $(k-1)^{\text{th}}$ and k^{th} successful transmissions made by source i. Then,

$$\begin{aligned} \Delta_i &= \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} A_i[t] \\ &\overset{(a)}{=} \Delta_i^{\text{TA}} + \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} A_i^s\left[\max\{U_k^{(i)} : U_k^{(i)} < t\}\right] \\ &= \Delta_i^{\text{TA}} + \lim_{K\to\infty} \frac{\sum_{k=1}^{K}\sum_{l=1}^{I_i[k]} A_i^s[U_k^{(i)}]}{\sum_{k=1}^{K} I_i[k]} \\ &= \Delta_i^{\text{TA}} + \lim_{K\to\infty} \frac{\sum_{k=1}^{K} A_i^s[U_k^{(i)}] I_i[k]}{\sum_{k=1}^{K} I_i[k]} \\ &= \Delta_i^{\text{TA}} + \frac{\mathbb{E}\left[A_i^s[U_k^{(i)}] I_i[k]\right]}{\mathbb{E}\left[I_i[k]\right]} \overset{(b)}{=} \Delta_i^{\text{TA}} + \mathbb{E}[A_i^s] \end{aligned}$$

where (a) follows from (2.59) and (2.60), and (b) follows from the independence between transmission policy and arrival processes. Average age of the packet at the source is $\mathbb{E}[A_i^s]$ and is equal to $1/\lambda_i$ [26]. The optimal value of Δ_i^{TA} was shown to be asymptotically $1.4169n$ while $1/\lambda_i$ diminishes compared to Δ_i^{TA}, since $\lim_{n\to\infty} \frac{1/\lambda_i}{n} =$

0. As a result, optimal age can be upper bounded by $1.4169n$ in the limit of infinite n since this average age is asymptotically achievable by the modified threshold-ALOHA policy where the policy is worsened by forcing sources to make a transmission attempt when they don't have a fresh packet available. On the other hand, optimal age is lower bounded by $1.4169n$ as well since having a fresh packet available to send at all times is guaranteed to not increase the average age. Hence,

$$\lim_{n\to\infty} \frac{\Delta^*}{n} = 1.4169. \tag{2.61}$$

2.3 Numerical Results and Discussion

In this section, we present numerical plots and simulation results to illustrate our theoretical findings and to perform comparisons with related policies. In Fig. 2.6, optimal AoI results can be observed under threshold-ALOHA, slotted ALOHA and stationary age-based thinning (SAT) policy presented in [1]. Simulations of SAT and threshold-ALOHA were performed under different n values ranging from 50 to 1000 and run for 10^7 time slots. Initial states of the users were randomized so that a bias from the initial congestion of having too many active users could be prevented and the decentralized structure of the algorithm could be preserved. Note that avg. AoI of threshold-ALOHA rises with slope 1.4169 with network size which is almost the same as SAT and roughly half the slope of slotted ALOHA. Besides, the two recent studies [1] and [14] stand out as closely related to our work. Below, we clarify the contributions in this study in the light of these two related studies:

An independent analysis of threshold-ALOHA was carried out in [14], and the results were supported by hardware experiments in [27]. The analysis in [14], however, is based on an approximation that the states of the sources are independent of each other. This stands in contrast to the results of our steady-state analysis (Lemma 1) which identifies a strong dependence between the states of the sources through the number of active sources in the system. Moreover, the analysis in [14] was limited to the case of the transmission probability, τ, being below $\frac{2}{n}$, which, as we show in this paper, is quite far from the optimal choice of the transmission probability, $\frac{4.69}{n}$. This is consistent with the simulation results presented in [14] that indicate an AoI ($2n$, or $1.6n$ in two different simulation plots), which are above the optimal value of $1.41n$.

	TA	SAT [1]
Tx	0.9	e^{-1}

	TA	SAT [1]
Rx	0.9	n

Table 2.2: Comparison of the expected number of users in Tx and Rx modes in a time slot during steady-state in threshold-ALOHA and SAT [1] under optimal parameters.

Akin to threshold-ALOHA (TA), SAT dictates that users stay silent before their ages reach a fixed threshold. However, unlike TA, the probability of an active user making a transmission is not fixed. Each user computes its transmission probability according to its estimate of the number of active users. Users keep their estimates up-to-date by staying in receive mode to detect collisions, even when they are not active. In TA, on the other hand, users need to listen for ACK/NACK feedback only after their own transmission attempts, which would allow them to go to an idle or sleep mode when they are inactive. This may lead to a major difference between the power consumption needed to implement each policy. As it can be seen in Table 1, the number of users in receive mode in each slot increases linearly with n in the SAT policy, whereas it is constant for TA. The constant value of 0.9 originates from the function G described in Table 2, defined as the average number of users that make a transmission attempt per time slot. At moderate transmission radii (*e.g.*, below 100 meters), typical in IoT and sensor networks, the power consumption in receive mode is comparable to that in transmit mode. Therefore, as network density increases, the Rx energy consumption is likely to be dominant [28]. This suggests that TA may be more suitable to dense IoT deployments with energy constrained nodes.

The extensive analysis in [1] has shown that with SAT the average AoI scales as $\frac{e}{2}n$ $(1.3591n)$. We exhibit in this work that TA is able to achieve a scaling of $1.4169n$. In other words, SAT asymptotically achieves a 4% age advantage over TA. In terms of throughput, the two fare closely: both policies achieve a throughput close to the slotted ALOHA limit, which is around e^{-1}. We finally remark that the 4% advantage achieved by SAT comes at a cost of a considerably increased feedback requirement, power consumption and computational complexity.

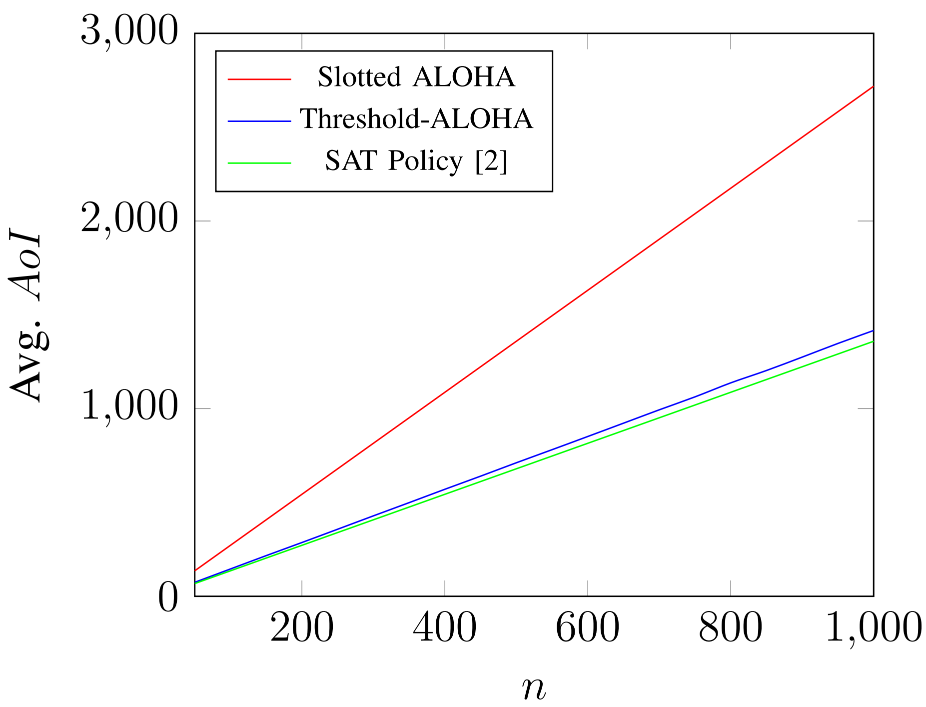


Figure 2.6: Optimal time average AoI vs n, number of sources, under Slotted ALOHA (computed from (2.54)), threshold-ALOHA (simulated) and SAT Policy (simulated)

We showed above that threshold-ALOHA keeps the number of active users at any time at steady state at about *one-fifth of all users* (see Table 2.1), with optimal parameter settings. This enables the users to utilize the channel more efficiently, approaching throughput of e^{-1} packets per slot. Fig. 2.7, plots Ge^{-G}, where $G = 1$ has been marked as the throughput optimal operating point of ordinary slotted ALOHA and $G = 0.89$ has been marked for threshold-ALOHA. The corresponding throughput values are $e^{-1}$and 0.3658, respectively, which differ by less than 1%. Hence, *threshold-ALOHA nearly halves avg. AoI while maintaining a near-optimal throughput.*

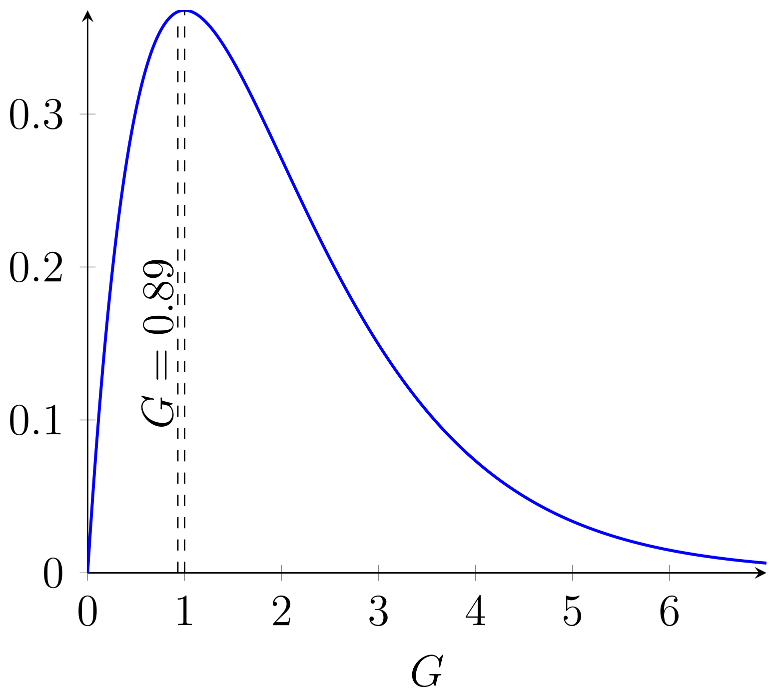


Figure 2.7: Throughput vs the expected number of transmissions in a slot, G.

CHAPTER 3

MUMISTA: MULTIPLE MINI SLOTTED THRESHOLD ALOHA

In the previous chapter, we have investigated the performance of threshold Aloha and despite the policy nearly halving the average AoI compared to slotted Aloha; it still suffers from the low throughput of e^{-1}.Further improvements on the information freshness are possible only with a dramatic improvement of the throughput [29]. With this intuition, we were motivated to create a new slot structure that would be able to utilize the channel to a greater degree whilst preserving the random access nature. Multiple Mini Slotted Threshold Aloha (MuMiSTA) is a reservation based policy that can achieve nearly optimal throughput and average AoI that is comparable to the round-robin policy. In the rest of this chapter, we provide a deep analysis of MuMiSTA policy and determine the throughput-optimality and AoI-optimality conditions.

3.1 System Model

We consider n nodes that interact with a common access point (AP) through a random access channel. The information packets are generated by the nodes immediately before a transmission takes place. The nodes perform transmissions in a synchronized manner. The time horizon is slotted, consisting of mini slots and data slots (explained in II-A). No collision resolution is performed at the destination; if two or more nodes attempt a transmission at the same time, all packets are discarded by the AP. Lost packets are not retransmitted, the nodes continue their random access policies with fresh packets in the next slot. We refer to the nodes that are ready to make a transmission and a possible contender for the random access channel as *active* nodes; whereas the remaining nodes are referred to as *passive* nodes. The classification of nodes are carried out according to the transmission policy.

In MuMiSTA, we employ a novel slot structure where a time slot consists of $W-1$ mini slots and a single data slot. In the beginning of the first mini slot, the active users transmit a packet to reserve the data slot with probability τ_1. If multiple nodes make a transmission attempt in the mini slot, there will be a collision and the AP will not be able to identify the transmitter. In this case, only the users that made a transmission attempt in the first mini slot may try again in the next mini slot with probability τ_2. This process will be repeated in the remaining mini slots until either $W-1$ mini slots have passed and or there was a mini slot in which there has been at most 1 user that has made a transmission attempt. If a node skips a mini slot by not making a transmission attempt, it yields the chance to transmit a packet in the data slot. If there is still a collision in the last mini slot after all the mini slots, all the remaining nodes may make a transmission attempt to send their data packet in the data slot with probability τ_W. In summary, there are W windows of oppurtunities to determine a single node that will successfully transmit a packet in the data slot. In the special case of $W=1$, there are no mini slots and the setup is identical to that of slotted ALOHA. The ratio between the length of a data slot and a mini slot is L ($L>1$).

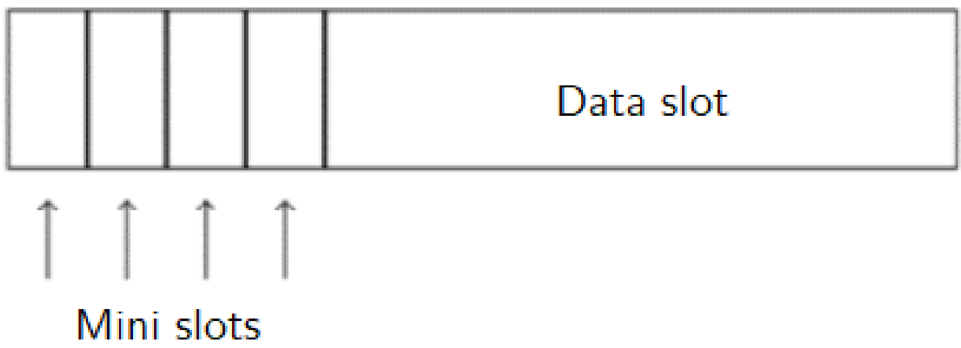


Figure 3.1: Diagram of a single time slot, consisting of multiple mini slots and a data slot

Similar to the Threshold-ALOHA policy of the previous chapter, we employ an age threshold of Γ on the nodes before they may become active. Such an age threshold reduces the number of contenders for the channel use increase the probability of a successful transmission by a node with a greater age.

3.2 Problem Definition and Analysis

3.2.1 Throughput Optimization Under The Mini Slot Extension

In this section, we derive a simplified expression for the throughput and calculate optimal parameters. A successful transmission happens when there is a single node that makes a transmission attempt in one of the mini slots. To denote the throughput, we use $T(n, \tau_1, \tau_2, ..., \tau_W)$, where n is the active node count and $\tau_1, \tau_2, ..., \tau_W$ are the transmission probabilities described in II-A.

3.2.1.1 Ideal case

We shall initially neglect the size of the mini slots so that the throughput of the data slots can be calculated. Later, we discuss the effect of L on the mini slots. The following theorem presents the throughput in this sense:

Theorem 5. *The throughput of MuMiSTA with n users for mini slots of negligible size is:*

$$T(n, \tau_1, \tau_2, ..., \tau_W) = n\left(1-\kappa_W\right)^{n-1}\kappa_W + \sum_{j=1}^{W-1} n\left(1-\kappa_j\right)^{n-1}\kappa_j\left(1-\tau_{j+1}\right) \quad (3.1)$$

where κ_j is defined as $\kappa_j \triangleq \prod_{i=1}^{j}\tau_i$.

Proof. We use proof by induction. The throughput is $T(n, \tau_1) = n\tau_1\left(1-\tau_1\right)^{n-1}$ for $W = 1$, a successful transmission occurs iff there is exactly one user making a transmission attempt.

Next, we assume the throughput expression holds for $W = w_0$ and we shall analyze over the case of $W = w_0 + 1$. To this end, we may derive an expression for the probability of successful transmission by conditioning on the number of users that make a transmission attempt in the first mini slot. If there is only 1 user making a transmission attempt, it will be a success and there will be no need for the succeeding mini slots. If there are $i > 1$ transmitters, the probability of success will be determined from the following mini slots. As a result,

$$T(n, \tau_1, \tau_2, ..., \tau_{w_0+1}) = n\tau_1(1-\tau_1)^{n-1} + \sum_{i=2}^{n} \binom{n}{i} \tau_1^i (1-\tau_1)^{n-i} T(i, \tau_2, ..., \tau_{w_0+1}) \tag{3.2}$$

Note that $T(i, \tau_2, ..., \tau_{w_0+1})$ corresponds to a system for which $W = w_0$ and therefore,

$$\begin{aligned} T(i, \tau_2, ..., \tau_{w_0+1}) &= i\tau_2...\tau_{w_0+1} \left(1 - \tau_2...\tau_{w_0+1}\right)^{i-1} \\ &\quad + \sum_{j=2}^{w_0} i\tau_2...\tau_j \left(1 - \tau_2...\tau_j\right)^{i-1} \left(1 - \tau_{j+1}\right) \\ &= i\left(1 - \frac{\kappa_{w_0+1}}{\tau_1}\right)^{i-1} \frac{\kappa_{w_0+1}}{\tau_1} + \sum_{j=2}^{w_0} i\left(1 - \frac{\kappa_j}{\tau_1}\right)^{i-1} \frac{\kappa_j}{\tau_1} \left(1 - \tau_{j+1}\right) \end{aligned} \tag{3.3}$$

due to the induction assumption.

$$\begin{aligned} T(n, \tau_1, \tau_2, ..., \tau_{w_0+1}) &= n\tau_1(1-\tau_1)^{n-1} + \sum_{i=2}^{n} \binom{n}{i} \tau_1^i (1-\tau_1)^{n-i} \Bigg[\\ &\quad i\left(1 - \frac{\kappa_{w_0+1}}{\tau_1}\right)^{i-1} \frac{\kappa_{w_0+1}}{\tau_1} + \sum_{j=2}^{w_0} i\left(1 - \frac{\kappa_j}{\tau_1}\right)^{i-1} \frac{\kappa_j}{\tau_1} \left(1 - \tau_{j+1}\right) \Bigg] \\ &= n\tau_1(1-\tau_1)^{n-1} + \sum_{i=2}^{n} \binom{n}{i} \tau_1^i (1-\tau_1)^{n-i} i \frac{\kappa_{w_0+1}}{\tau_1} \left(1 - \frac{\kappa_{w_0+1}}{\tau_1}\right)^{i-1} \\ &\quad + \sum_{j=2}^{w_0} \sum_{i=2}^{n} \binom{n}{i} \tau_1^i (1-\tau_1)^{n-i} i \left(1 - \frac{\kappa_j}{\tau_1}\right)^{i-1} \frac{\kappa_j}{\tau_1} \left(1 - \tau_{j+1}\right) \\ &\overset{(a)}{=} n\tau_1(1-\tau_1)^{n-1} + \sum_{i=1}^{n-1} n \binom{n-1}{i} \tau_1^i (1-\tau_1)^{n-1-i} \kappa_{w_0+1} \left(1 - \frac{\kappa_{w_0+1}}{\tau_1}\right)^{i} \\ &\quad + \sum_{j=2}^{w_0} \sum_{i=1}^{n-1} n\kappa_j (1 - \tau_{j+1}) \binom{n-1}{i} \tau_1^i (1-\tau_1)^{n-1-i} \left(1 - \frac{\kappa_j}{\tau_1}\right)^{i} \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&\overset{(b)}{=} n\tau_1(1-\tau_1)^{n-1} + n\kappa_{w_0+1}\left[(1-\kappa_{w_0+1})^{n-1} - (1-\tau_1)^{n-1}\right] \\
&\qquad + \sum_{j=2}^{w_0} n\kappa_j(1-\tau_{j+1})\left[(1-\kappa_j)^{n-1} - (1-\tau_1)^{n-1}\right] \\
&\overset{(c)}{=} n\tau_1(1-\tau_1)^{n-1} + n\kappa_{w_0+1}(1-\kappa_{w_0+1})^{n-1} \\
&\qquad + \sum_{j=2}^{w_0} n\kappa_j(1-\tau_{j+1})(1-\kappa_j)^{n-1} - n\tau_1\tau_2(1-\tau_1)^{n-1} \\
&= n\kappa_{w_0+1}(1-\kappa_{w_0+1})^{n-1} + \sum_{j=1}^{w_0} n\kappa_j(1-\tau_{j+1})(1-\kappa_j)^{n-1}
\end{aligned}
\tag{3.5}
$$

where (a) follows from $i\binom{n}{i} = n\binom{n-1}{i-1}$, (b) follows from the binomial sum and (c) follows from $\sum_{j=2}^{w_0} n\kappa_j(1-\tau_{j+1}) = \sum_{j=2}^{w_0} n\kappa_j - n\kappa_{j+1} = n\tau_1\tau_2 - n\kappa_{w_0+1}$. Induction is complete. □

Using the expression in Theorem 5, maximum achievable throughput can be calculated for finite number of users by calculating its partial derivatives. We will conduct our analysis for infinitely many users. We define G as the expected number of transmission attempts in the first mini slot, or alternatively,

$$\lim_{n\to\infty} n\tau_1 = G \tag{3.6}$$

Then, with a slight abuse of notation, the throughput expression can be modified to fit our problem in the infinite-user case:

$$T(G, \tau_2, ..., \tau_W) = \lim_{n\to\infty} T(n, \tau_1, \tau_2, ..., \tau_W) \tag{3.7}$$

Consequently,

$$
\begin{aligned}
T(G, \tau_2, ..., \tau_W) &= \zeta_W \exp(-\zeta_W) + \sum_{j=1}^{W-1} \zeta_j \exp(-\zeta_j)(1-\tau_{j+1}) \\
&= \zeta_W \exp(-\zeta_W) + \sum_{j=1}^{W-1} (\zeta_j - \zeta_{j+1}) \exp(-\zeta_j)
\end{aligned}
\tag{3.8}
$$

follows from Theorem 5, where ζ_j is defined as:

$$\zeta_j \triangleq \lim_{n\to\infty} n\kappa_j = \lim_{n\to\infty} n\prod_{i=1}^{j}\tau_i = G\prod_{i=2}^{j}\tau_i \tag{3.9}$$

The following theorem establishes the throughput-optimal parameters along with the maximum achievable throughput.

Theorem 6. *Let $\{B_k\}_{k=1}^{\infty}$ sequence be defined as:*

$$B_1 = 1 \tag{3.10}$$

$$B_k = 1 - \exp(-B_{k-1}) \tag{3.11}$$

Maximum throughput for infinitely many users with W rounds of transmission attempts ($W-1$ mini slots) is

$$T_{max}^{(W)} = 1 - B_{W+1} \tag{3.12}$$

The optimal parameters are:

$$G^* = \sum_{i=1}^{W} B_i \tag{3.13}$$

$$\tau_j^* = \frac{\sum_{i=j}^{W} B_i}{\sum_{i=j-1}^{W} B_i}, \qquad j = 2, \ldots, W \tag{3.14}$$

Proof. In order to maximize the throughput, we shall set all the partial derivatives of the throughput to 0:

$$\frac{\partial T(G, \tau_2, ..., \tau_W)}{\partial G} = 0 \tag{3.15}$$

$$\frac{\partial T(G, \tau_2, ..., \tau_W)}{\partial \tau_j} = 0, \qquad j = 2, \ldots, W \tag{3.16}$$

From (3.9), it can be seen that:

$$\frac{\partial \zeta_j}{\partial \tau_i} = \frac{\zeta_j}{\tau_i}, \qquad W \geq j \geq i \geq 2 \tag{3.17}$$

and

$$\frac{\partial \zeta_j \exp(-\zeta_j)}{\partial \tau_i} = \frac{\zeta_j \exp(-\zeta_j)(1-\zeta_j)}{\tau_i}, \qquad W \geq j \geq i \geq 2 \tag{3.18}$$

If i is greater than j, ζ_j does not include a τ_i term and the derivative is equal to 0. By calculating the derivative over τ_W, we obtain:

$$\frac{\partial T(G, \tau_2, ..., \tau_W)}{\partial \tau_W} = \frac{\zeta_W \exp(-\zeta_W)(1-\zeta_W)}{\tau_W} - \exp(-\zeta_{W-1})\zeta_{W-1} = 0 \tag{3.19}$$

Noting that $\zeta_W = \zeta_{W-1}\tau_W$, we get:

$$1 - \zeta_W = \exp(\zeta_W - \zeta_{W-1}). \tag{3.20}$$

For $i = 2, 3, \ldots, W-1$, we may write the partial derivative from (3.9) as:

$$\begin{aligned}\frac{\partial T(G, \tau_2, ..., \tau_W)}{\partial \tau_i} &= \frac{\zeta_W \exp(-\zeta_W)(1-\zeta_W)}{\tau_i} \\ &\quad + \sum_{j=i}^{W-1} \frac{\zeta_j \exp(-\zeta_j)(1-\zeta_j)}{\tau_i}(1-\tau_{j+1}) - \exp(-\zeta_{i-1})\zeta_{i-1} = 0\end{aligned} \tag{3.21}$$

Then,

$$\begin{aligned}\tau_i \frac{\partial T(G, \tau_2, ..., \tau_W)}{\partial \tau_i} - \tau_{i+1} \frac{\partial T(G, \tau_2, ..., \tau_W)}{\partial \tau_{i+1}} &= \zeta_i \exp(-\zeta_j)(1-\zeta_j)(1-\tau_{i+1}) \\ &\quad - \exp(-\zeta_{i-1})\zeta_i + \exp(-\zeta_i)\zeta_{i+1} = 0\end{aligned} \tag{3.22}$$

Above can be rewritten as:

$$(1 - \zeta_i + \zeta_{i+1}) = \exp(\zeta_i - \zeta_{i-1}) \tag{3.23}$$

which holds for $i = 2, 3, \ldots, W-1$.

Lastly, $G \partial T(G, \tau_2, ..., \tau_W)/\partial G - \tau_2 \partial T(G, \tau_2, ..., \tau_W)/\partial \tau_2 = 0$ yields:

$$\exp(-\zeta_1)\,(1 - \zeta_1 + \zeta_2) = 0 \tag{3.24}$$

At this point, we can see that $B_i = \zeta_i - \zeta_{i+1}$, with (3.24) and (3.23) leading to (3.10) and (3.11), respectively. Accordingly, (3.20) becomes:

$$1 - \zeta_W = \exp(-B_{W-1}) = 1 - B_W. \tag{3.25}$$

and $\zeta_W = B_W$. Then, $\zeta_i = \sum_{j=i}^{W} B_j$. We can write the parameters in terms of ζ_j's as $G = \zeta_1$ and $\tau_i = \zeta_{i+1}/\zeta_i$, from which (3.13) and (3.14) follows. Finally, we evaluate the throughput as:

$$\begin{aligned}T(G, \tau_2, ..., \tau_W) &= \zeta_W \exp(-\zeta_W) + \sum_{j=1}^{W-1} (\zeta_j - \zeta_{j+1}) \exp(-\zeta_j) \\ &= B_W \exp(-B_W) + \sum_{j=1}^{W-1} B_j \exp\left(-\sum_{i=j}^{W} B_i\right) \\ &= B_W \exp(-B_W) + \sum_{j=1}^{W-1} B_j \prod_{i=j}^{W} \exp(-B_i) \\ &= B_W(1 - B_{W+1}) + \sum_{j=1}^{W-1} B_j \prod_{i=j}^{W} (1 - B_{i+1}) \\ &= 1 - B_{W+1}\end{aligned} \tag{3.26}$$

□

The maximum throughput is expressed with an iterative sequence, and converges to 1 as the number of mini slots increases. In the following, we describe the asymptotic behaviour of B_k sequence in order to provide an approximation for the use cases with large W values.

Lemma 3. *i) The $\{B_k\}_{k=1}^{\infty}$ sequence above satisfies the following:*

$$\lim_{k\to\infty} kB_k = 2 \tag{3.27}$$

ii) Maximum throughput in MuMiSTA satisfies the following:

$$T_{max}^{(W)} = 1 - \frac{2}{W} + o\left(\frac{1}{W}\right) \tag{3.28}$$

Proof. We define D_k sequence as follows:

$$D_k = \frac{1}{B_k} - \frac{1}{B_{k-1}} = \frac{1}{1 - \exp(-B_{k-1})} - \frac{1}{B_{k-1}}. \tag{3.29}$$

Alternatively, D_k can be expressed as a function of B_{k-1}:

$$D_k = g(B_{k-1}), \tag{3.30}$$

where g function is defined as:

$$g(x) = \frac{1}{1 - \exp(-x)} - \frac{1}{x} = \frac{x - 1 + \exp(-x)}{x - x\exp(-x)}. \tag{3.31}$$

Further, the limit of g as x (or B_{k-1}) goes to 0 is $1/2$, found from the L'Hôpital's rule.

$$\lim_{x\to 0} g(x) = \frac{1}{2}. \tag{3.32}$$

$$\lim_{k\to\infty} D_k = \lim_{k\to\infty} \frac{1}{B_k} - \frac{1}{B_{k-1}} = \frac{1}{2}. \tag{3.33}$$

As a result, Stolz–Cesàro theorem can be used to derive the following limit:

$$\lim_{k\to\infty} \frac{1}{kB_k} = 2, \tag{3.34}$$

which is equivalent to the first part of the lemma. The second part then follows immediately from Theorem 6. □

For the special case of $W = 1$, there are no mini slots and the policy is equal to slotted Aloha. The results of the theorem are consistent for this case; the maximum throughput is e^{-1} and ideal G is equal to 1.

3.2.1.2 Realistic case

In a realistic scenario, the length of the mini slots shall be taken under consideration when calculating the throughput. The data slot spans $\frac{L}{L+W-1}$ of a time slot in total. The throughput in this case can simply be calculated by multiplying the throughput from $L = \infty$ case with $\frac{L}{L+W-1}$. Hence,

$$T_{max}^{(L,W)} = \frac{L(1 - B_{W+1})}{L + W - 1} \tag{3.35}$$

Ideal value of W can be found by finding W^* such that:

$$T_{max}^{(L,W^*)} = \max\left\{T_{max}^{(L,W^*-1)}, T_{max}^{(L,W^*)}, T_{max}^{(L,W^*+1)}\right\}. \tag{3.36}$$

Theorem 7. *Throughput optimal window size W^* is the greatest integer such that $h(W^*) \leq L$ where $h(w)$ is defined as:*

$$h(w) = \frac{1}{1 - \exp(B_w - B_{w-1})} - (w - 1) \tag{3.37}$$

Proof. To prove the theorem, we study the $T_{max}^{(L,w-1)} \leq T_{max}^{(L,w)}$ statement. This is equivalent to:

$$\frac{L(1 - B_w)}{L + w - 2} \leq \frac{L(1 - B_{w+1})}{L + w - 1} \tag{3.38}$$

We use the property of (3.11) to obtain:

$$1 + \frac{1}{L + w - 2} = \frac{L + w - 1}{L + w - 2} \leq \frac{1 - B_{w+1}}{1 - B_w} = \exp(B_{w-1} - B_w) \tag{3.39}$$

Finally, above can be rewritten as:

$$\frac{1}{1 - \exp(B_w - B_{w-1})} - (w - 1) \leq L \tag{3.40}$$

The inequality in (3.40) should hold for $w = W^*$ but not for $w = W^* + 1$. □

The h sequence allows us to properly tune W to the size of the data slot and the mini slots. For large L values, optimal W can simply be approximated as $\sqrt{2L}$. Then the maximum achievable throughput then can be approximated to $1 - 2\sqrt{\frac{2}{L}}$.

3.2.2 AoI Optimization

3.2.2.1 Truncated State Space Model

In this section, we analyze the steady state behaviour of the MuMiSTA policy. We use the previously explained *truncated* state space method in our analysis. The truncated state vector is defined as:

$$\mathbf{A}^{\Gamma}[t] \triangleq \left\langle A_1^{\Gamma}[t] \quad A_2^{\Gamma}[t] \quad \dots \quad A_n^{\Gamma}[t] \right\rangle \tag{3.41}$$

where $A_i^{\Gamma}[t] \in \{1, 2, \dots, \Gamma\}$ is the AoI of source *i* at time $t \in \mathbb{Z}^+$ truncated at Γ and evolves as:

$$A_i^{\Gamma}[t] = \begin{cases} 1, & \text{source } i \text{ updates at time } t-1, \\ \min\left\{A_i^{\Gamma}[t] + 1,\ \Gamma\right\}, & \text{otherwise.} \end{cases} \tag{3.42}$$

Let S_m be the probability of a successful transmission happening in a time slot when there are m active users. Due to the symmetry and memorylessness, this is equivalent to the throughput of a network with m nodes and no age-aware structure, *i.e.* $S_m = T(m, \tau_1, \tau_2, ..., \tau_W)$. In the following Lemma, we establish the distribution of the number of active users.

Lemma 4. *The truncated MC* $\{\mathbf{A}^{\Gamma}[t], t \geq 1\}$ *has the following properties:*

i) Given a state vector $\langle s_1 \quad s_2 \quad \dots \quad s_n \rangle$*, its steady state probability depends only on the number of entries that are equal to* Γ.

ii) Let P_m *be the total steady state probability of states having* m *active users. Then*

$$\frac{P_m}{P_{m-1}} = \frac{(1 - S_{m-1})(n - m + 1)}{S_m(\Gamma - 1 - n + m)}$$

iii) The steady state probability of having no active sources is

$$P_0 = \frac{1}{1 + \sum_{m=1}^{n} \prod_{i=1}^{m} \frac{(1-S_{i-1})(n-i+1)}{S_i(\Gamma-1-n+i)}}$$

Proof. We use the type argument of Lemma 1 in this proof. The type sets are:

- $\mathcal{T}_0 \triangleq (M-1, \{u_1, u_2, \dots, u_{n-M}, 1\})$

- $\mathcal{T}_1 \triangleq (M, \{u_1, u_2, \ldots, u_{n-M}\})$
- $\mathcal{T}_2 \triangleq (M, \{u_1 - 1, u_2 - 1, \ldots, u_{n-M} - 1\})$
- $\mathcal{T}_3 \triangleq (M - 1, \{\Gamma - 1, u_1 - 1, u_2 - 1, \ldots, u_{n-M} - 1\})$

States of types $\mathcal{T}_0$ and $\mathcal{T}_1$ are preceded by the states of types $\mathcal{T}_2$ and $\mathcal{T}_3$ only. Furthermore, states of types $\mathcal{T}_2$ and $\mathcal{T}_3$ are succeeded by states of types $\mathcal{T}_0$ and $\mathcal{T}_1$ only. Steady state equations for states of types $\mathcal{T}_0$ and $\mathcal{T}_1$ can be written in the following way:

$$\pi_{\mathcal{T}_1} = \pi_{\mathcal{T}_2}(1 - MS_M) + \pi_{\mathcal{T}_3}M(1 - (M-1)S_{M-1}) \tag{3.43}$$

$$\pi_{\mathcal{T}_0} = \pi_{\mathcal{T}_2}S_M + \pi_{\mathcal{T}_3}(M-1)S_{M-1} \tag{3.44}$$

We next test the first property of the Lemma by assigning π_m to be the steady state probability of a state with m active users. By replacing $\pi_{\mathcal{T}_1}$ and $\pi_{\mathcal{T}_2}$ with π_M and $\pi_{\mathcal{T}_0}$ and $\pi_{\mathcal{T}_3}$ with π_{M-1} yields:

$$\pi_M = \pi_M(1 - MS_M) + \pi_{M-1}M(1 - (M-1)S_{M-1}), \tag{3.45}$$

$$\pi_{M-1} = \pi_M S_M + \pi_{M-1}(M-1)S_{M-1}. \tag{3.46}$$

These equations are reduced to the same equation below:

$$\frac{\pi_m}{\pi_{m-1}} = \frac{1 - (m-1)S_{m-1}}{S_m}. \tag{3.47}$$

Eq. (3.47) gives the only solution set due to the uniqueness of the steady state solution. Second and third properties follows accordingly. The total number of states corresponding to π_m are the number of recurrent system states with m sources at truncated age Γ:

$$N_m = \binom{n}{m}\frac{(\Gamma-1)!}{(\Gamma - n - 1 + m)!} \tag{3.48}$$

Recall that P_m was defined as the total probability of all states with m active sources. From the first property, each of these states are equiprobable with steady state probability π_m. Hence,

$$P_m = N_m \pi_m \tag{3.49}$$

$$\frac{P_m}{P_{m-1}} = \frac{(1-(m-1)S_{m-1})(n-m+1)}{mS_m(\Gamma-1-n+m)} \tag{3.50}$$

$$\sum_{m=0}^{N} P_m = 1 \tag{3.51}$$

From (3.50) and (3.51),

$$P_0(1+\sum_{m=1}^{n}\prod_{i=1}^{m}\frac{(1-(i-1)S_{i-1})(n-i+1)}{iS_i(\Gamma-1-n+i)}) = 1 \tag{3.52}$$

$$P_m = P_0\prod_{i=1}^{m}\frac{(1-(i-1)S_{i-1})(n-i+1)}{iS_i(\Gamma-1-n+i)} \tag{3.53}$$

provides the steady state solution. □

Corollary 2. *It should also be noted that* $mS_m = T(m, \tau_1, \tau_2, ..., \tau_W)$ *as the throughput is the success probability of all active users. Consequently,*

$$\frac{P_m}{P_{m-1}} = \frac{1-T(m-1,\tau_1,\tau_2,...,\tau_W)}{T(m,\tau_1,\tau_2,...,\tau_W)}\frac{n-m+1}{\Gamma-1-n+m} \tag{3.54}$$

3.2.2.2 Pivoted MC

The following theorem is analogous to Theorem 1 of the previous chapter. Notice that the theorem is a generalization of Theorem 1 and substituting $T(\alpha x, \tau_2, ..., \tau_W)$ with $x\alpha\exp(-x\alpha)$ grants the same expression as in Theorem 1.

Theorem 8. *For some* $r, \alpha \in \mathbb{R}^+$*, such that* $\lim_{n\to\infty}\frac{\Gamma}{n} = r$ *and* $\lim_{n\to\infty}\tau_1 n = \alpha$*, define* $f : (0,1) \to \mathbb{R}$*:*

$$f(x) = \ln\left(\frac{1}{T(\alpha x,\tau_2,...,\tau_W)} - 1\right) + \ln\left(\frac{r}{x+r-1} - 1\right) \tag{3.55}$$

Then, for all m *such that* $\lim_{n\to\infty}\frac{m}{n} = k \in (0,1)$ *and* $s \in \mathbb{Z}^+$

$$\lim_{n\to\infty}\ln\left(\frac{P_m^{(s)}}{P_{m-1}^{(s)}}\right) = f(k) \tag{3.56}$$

where $P_m^{(s)}$ *is the steady state probability of having* m *active sources (excluding the pivot source), given that state of the pivot source is* s*.*

Proof. See Appendix D. □

3.2.2.3 Large Network Asymptotics

As in the previous chapter, the local maxima of the PMF of active nodes are the roots of f where f is decreasing. Since $f(0^+) = \infty$ and $f(1^-) = -\infty$, therefore, such a root definitely exists. In the previous chapter, the number of such roots was shown to be either 1 or 2. Theorems 2 and 3 had shown that if f has 1 or 2 such roots, then the ratio of active users will converge to some k_0 that could be estimated. The discussion of these theorems hold for the generic case of MuMiSTA as well.

Lemma 5. *Let ξ, the **dominant root** of f, be defined as:*

$$\xi = k_u, \quad \{u\} = \arg\max_i \int_{k_0}^{k_i} f(k)dk \tag{3.57}$$

Then, for the sequence $\epsilon_n = cn^{-1/3}$ where $c \in \mathbb{R}^+$,

$$\Pr\left(\left|\frac{m}{n} - \xi\right| < \epsilon_n\right) \to 1 \tag{3.58}$$

Proof. The proof follows as in Appendix C, where k_0 is replaced with ζ. □

Finally, we present the average AoI expression that is derived as a result of Lemma 5. The result is analogous to Theorem 4.

Theorem 9. *For some $r, \alpha \in \mathbb{R}^+$, such that $\lim_{n\to\infty} \frac{\Gamma}{n} = r$ and $\lim_{n\to\infty} \tau_1 n = \alpha$, and let ξ be the dominant root of f. Then, the average AoI converges as:*

$$\lim_{n\to\infty} \frac{\Delta}{n} = \frac{\xi^2 + 1}{2T(\xi\alpha, \tau_2, \ldots, \tau_W)} \tag{3.59}$$

or alternatively,

$$\lim_{n\to\infty} \frac{\Delta}{n} = \frac{r}{2}\frac{1 + \xi^2}{1 - \xi} \tag{3.60}$$

3.2.3 Numerical Results and Discussion

Numerical optimization of the average AoI can be performed by searching through all of the system parameters. There are a total of $W + 1$ parameters that can be tuned; r and attempt probabilities, α, τ_2, …, τ_W. Searching through all of these parameters, however, is tedious with negligible benefit if it is not carried out inside throughput-optimal region. Hence, we derive and fix the values of attempt probabilities except for the very first mini slot from Theorem 6. As a result of this, there remains two parameters used to optimize AoI; r and α, as in threshold ALOHA.

The following optimization analysis and the simulations were performed for $W = 32$ case. The simulations were performed for $n = 10^4$ users and 10^6 time slots.

The throughput optimal value of G, or $\zeta\alpha$ as in Theorem 9, is 5.91 from Theorem 6. We have found through our computer-aided analysis that optimal AoI is achieved for $\alpha = 219$ or equivalently $\zeta = 0.0261$. For this set of values, the corresponding r was found to be 1.033. The channel throughput and average AoI are found to be 0.95 and $0.53062n$ respectively. It should be noted that an ideal round-robin policy that serves each user in a perfect order would achieve a perfect throughput of 1 and average AoI of $0.5n$. Hence, there is a loss of around 6% in terms of throughput and average AoI despite maintaining a random-access structure. Further, very low value of ζ indicates that only 2.6% of all users are active at any time slot a great majority of users sleep and conserve energy. Compared to slotted ALOHA policy, the throughput is improved by 158% and average AoI is reduced by 80%.

Hence, MuMiSTA grants great improvements to the performance of a homogenous random access setting in terms of throughput, information freshness and energy consumption. These improvements, however, come at the cost of a more complicated slot structure compared to slotted ALOHA. Consequently, depending on the system requirements, MuMiSTA becomes a favorable alternative to the traditional random access schemes in the future of IoT and mMTC applications.

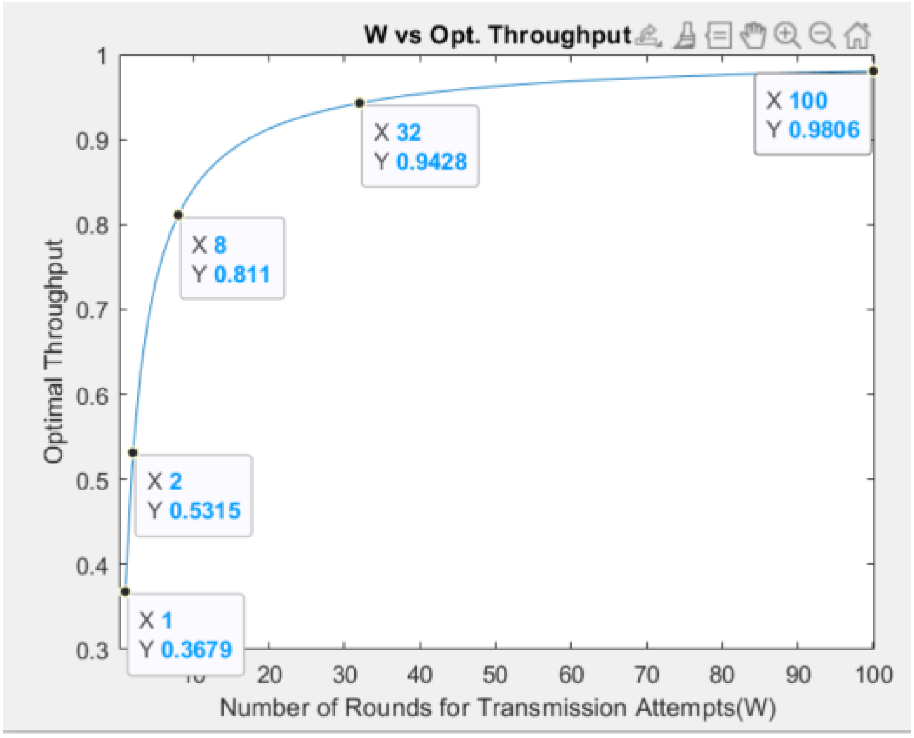


Figure 3.2: Optimal Throughput vs W

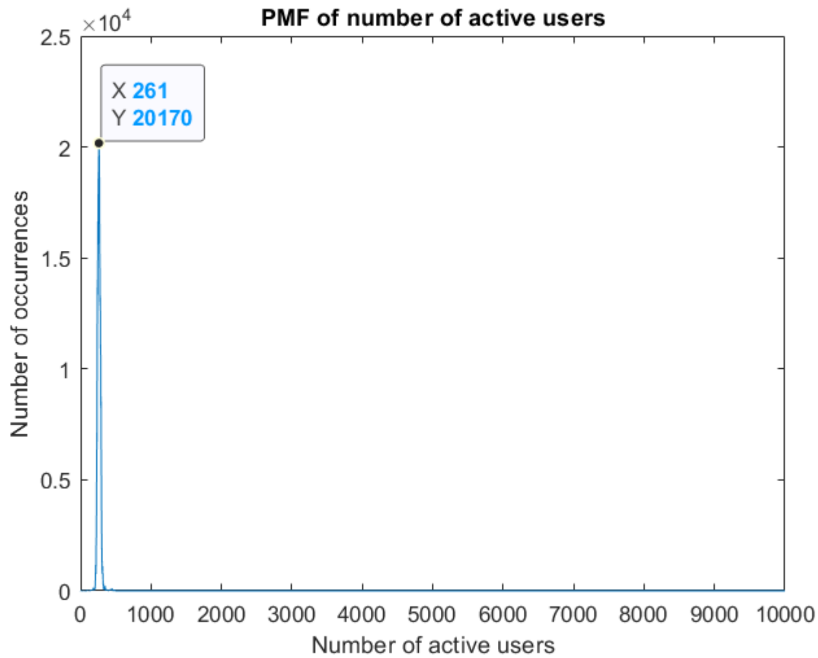


Figure 3.3: PMF of number of active users ($W = 32, \alpha = 219, r = 1.033$)

CHAPTER 4

OPTIMAL AGE PENALTY IN WIRELESS ENERGY TRANSFER

Wireless energy transfer (WET) using time-varying electric, magnetic, or electromagnetic fields, has been considered viable for various communication systems [30, 31, 32]. This technology is especially attractive in the scenario of collecting data from a sensor that does not have a significant energy source of its own. This implies a mode of energy harvesting, where the source of energy is directly controlled by the node who will pull the data from the sensor. The optimal planning of transmissions in such a scenario has been considered in the literature (see, e.g., [33], and references therein). In this paper, we formulate the problem from an Age of Information (AoI) optimization perspective.

Optimal transmission scheduling [34] is the problem of modifying rate and power in time according to energy availability, data demand, and channel variations, to transfer data as efficiently as possible, i.e., maximize the throughput with the given amount of energy, satisfy certain delay constraints, etc. Transmission scheduling to optimize AoI is relatively new.

In recent years, there has been a growing interest in the combined analysis of information freshness and energy harvesting. In [35], the problem of when to generate updates under detailed energy harvesting constraints (energy causality constraints) was formulated and solved. Each transmission consumes unit energy, and the goal of the transmitter is to spread its transmissions as evenly as possible in time, while respecting the energy causality constraints, to minimize average age. In [36], a stationary transmission policy was considered for a source that harvests energy at a constant rate λ, and has an infinite battery so that the energy causality constraints are not binding: there is always energy available when the source decides to transmit, as long as it does not use energy at higher rate than λ. In this model, packets are subject to iid delays Y in the channel. A single-server policy is maintained, hence a new packet can only start transmission once the transmission of the current packet is completed. Loosely speaking, then, the long term average rate of transmissions cannot exceed $1/E[Y]$. Consequently, if the expected delay, $E[Y]$, exceeds $1/\lambda$, then the transmitter will have to use energy at a rate lower than the rate it is harvested at. Otherwise, the transmitter has a choice to transmit at rate up to λ. The *β-optimal* policy proposed in [36] computes the policy that ensures that the arriving energy is used at rate λ whenever feasible. However, it notices the curious phenomenon that with this policy, the resulting age is non-monotone in the energy harvest rate λ. This indicates that the policy is not optimal in general. The optimal policy was shown in [37] to be one that possibly inserts a non-zero waiting time, $Z(Y)$, depending on the value of delay, even though $E[Y + Z(Y)] > 1/\lambda$. In other words, for many delay distributions, it is not optimal in terms of average age to transmit at the largest allowed update rate. The result was also generalized to Markovian delay processes and general age penalties in [37].

In this chapter, we consider a model where the receiver pulls data from a sensor by sending energy to be harvested by a transmitter connected to that sensor (see Fig. 4.1). This model allows the receiver to optimize the amount of energy it will deliver to the transmitter by taking the channel state information into account and thus, to control the long-term average age of information (AoI). Our aim is to derive the average age penalty in closed form and minimize it. By formulating minimization as a constrained Markov decision problem, we obtain an optimal decision policy which has minimum energy consumption and keeps the flow retrieved from the sensor as fresh as possible.

The problem we study is closely related to the above literature [36, 37] in the sense that it includes a finite average energy constraint, and a delay process that is caused by the channel state being on or off: when the transmitter makes a decision to transmit, the update is immediately received, followed by a random number of "off" slots during which there is no opportunity to pull data and the age increases. On the other hand, it differs from the models in [36, 37] in the sense that successful transmissions reset the age down to a deterministic constant. This implies zero-wait being optimal whenever feasible, hence the interesting case to be analyzed for this problem is the regime where zero-wait is not feasible.

There have been studies on transmission scheduling under WET constraints: In [38], time average AoI is investigated in a WET system with a Rayleigh block fading channel, where the transmitter waits until its battery is completely filled and uses all the acquired energy for a single transmission. In [39], an energy harvesting transmitter with a finite battery is studied in continuous time and it is shown that a threshold policy optimizes the expected time average age penalty. Another closely related recent work is [40], which studies the long-term time average AoI under a constraint on the average number of transmissions at the source node and examines standard ARQ and hybrid ARQ (HARQ) protocols. Threshold policies for controlling age under various energy harvesting settings have been studied in recent literature (see [41], and references therein.)

4.1 System Model and Problem Formulation

A point-to-point channel comprising a transmitter-receiver pair is considered. The channel can be in one of two states during any time slot: G (good) or B (bad). The transmitter is passive, and the receiver will decide when to send energy to the transmitter, and pull data from it. The transmitter, which relies solely on the energy harvested from the receiver, is only responsible for transmitting data to the receiver on demand, and each transmission takes one time slot duration and requires one unit of energy. It is also assumed that the receiver has an infinite battery, hence the energy causality constraints are inactive as in [36]. The allowed long term average energy usage is constrained by λ units per time slot. Transmissions always fail while the channel state is B and succeed otherwise. The random transitions of the channel states are analyzed under two models: (i) IID in each time slot, and (ii) Markovian.

Whenever the receiver receives a new packet from the source, it resets the age to unity at the end of the slot. In the absence of a new reception, the age increases by 1 with every new slot. Consequently, the age at the end of time slot t, denoted by Δ_t, is known by the receiver. The state of the system at any time t can be described by the AoI Δ_t and the channel state C_t at that time.

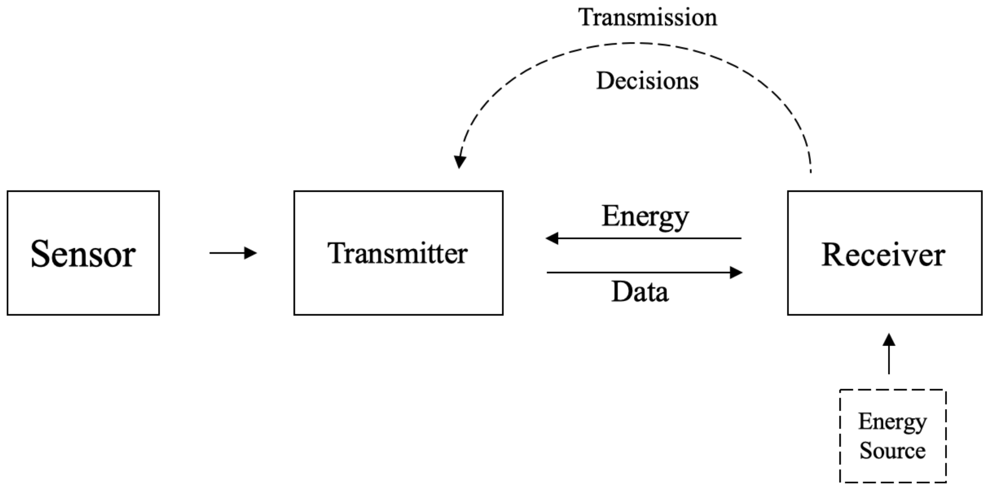


Figure 4.1: System model.

In order to generically quantify the staleness of data packets under different conditions, we define a general age penalty function $g(\Delta)$ as a function of AoI. The function $g : \mathbb{Z}^+ \to \mathbb{R}$ is non-decreasing. In the rest of this paper, we analyze the time average age penalty function. If the age penalty function is an identity function, the expected age penalty becomes the time average AoI and if $g(\Delta) = \mathbb{1}_{\Delta>\gamma}$, then expected age penalty becomes the age violation probability; corresponding to two commonly used metrics in the literature.

4.2 Problem Definition and Analysis

This leads to the constrained Markov decision process (CMDP) [42] formulation, defined by the 5-tuple: $(\mathcal{S}, \mathcal{U}, P, c, d)$ with the countable set of states $\mathcal{S} = \mathbb{Z}^+ \times \{\mathrm{G}, \mathrm{B}\}$ and the finite action set $\mathcal{U} = \{0, 1\}$. $u_t = 1$ denotes that the transmission will be performed and $u_t = 0$ denotes that no transmission occurs. The state s_t consists of the age Δ_t and the channel state C_t at time t. P refers to the transition function, where P$(s'|s, u) = Pr(s_{t+1} = s'|s_t = s, u_t = u)$ is the probability that action u in state s at time t will lead to state s at time $t + 1$. The cost function $c : \mathcal{S} \times \mathcal{U} \to \mathbb{R}$ is a non-decreasing function of the AoI at the destination, and is defined as $c(s, u) = g(\Delta_t)$, for any $s \in \mathcal{S}$, $a \in$ A, independently of action a. The transmission cost d is related with the energy constraint λ and is identical for each transmission, $d = 1$ if $u_t = 1$ and $d = 0$ otherwise. The age Δ_t evolves as:

$$\Delta_{t+1} = \begin{cases} 1, & \text{if } C_t = \text{G and } u_t = 1 \\ \Delta_t + 1, & \text{otherwise} \end{cases} \tag{4.1}$$

The evolution of the channel states is examined over two different scenarios. In Section 4.2.1, we assume that the channel state becomes G and B at each time slot in an independent and identically distributed (IID) fashion with their corresponding probability values P_G and P_B.

$$\Pr(C_{t+1} = c) = \begin{cases} P_G, & \text{if } c = \text{G} \\ P_B, & \text{if } c = \text{B} \end{cases} \tag{4.2}$$

where $P_G > 0$ and C_{t+1} is independent of the age or the past realizations of the channel states.

In Section 4.2.2, the results obtained for IID channel states are extended by considering time-correlated channel states which evolve as a Markovian process:

$$\Pr(C_{t+1} = c_1 \mid C_t = c_0) = \begin{cases} 1 - p_{10}, & \text{if } (c_1, c_0) = \text{(G,G)} \\ p_{01}, & \text{if } (c_1, c_0) = \text{(G,B)} \\ p_{10}, & \text{if } (c_1, c_0) = \text{(B,G)} \\ 1 - p_{01}, & \text{if } (c_1, c_0) = \text{(B,B)} \end{cases} \tag{4.3}$$

where $p_{ij} \in (0, 1)$, with $i, j \in \{0, 1\}$ indices standing for the channel state in former and latter time slots, respectively.

A stationary policy is a decision rule denoted by $\pi : \mathcal{S} \times \mathcal{U} \rightarrow [0, 1]$ which maps the states s into actions a with some probability $\pi(u|s)$. We try to minimize the average age penalty under energy constraint λ, given the initial state $s_0 = (1, \text{G})$. In this manner, our focus is the age penalty function $g(\Delta_t)$. We can state the CMDP optimization problem as follows, where $E[\cdot]$ represents expectation with respect to the distribution of the age process induced by policy π and channel states C_t:

$$\begin{aligned} \min_{\pi} \Delta^{\pi}(s_0) = \quad & \lim_{T\to\infty} \frac{1}{T} E[\sum_{t=1}^{T} g(\Delta_t)|s_0], \\ \text{s.t.} \quad & \lim_{T\to\infty} \frac{1}{T} E[\sum_{t=1}^{T} u_t^{\pi}|s_0] \leq \lambda. \end{aligned} \tag{4.4}$$

A policy π^* that is a solution of the above minimization problem is called optimal, and we are interested in finding optimal policies.

4.2.1 Memoryless Channel

Constrained MDPs with countably infinite state-spaces as defined in Problem 4.2 are generally difficult to solve since a stationary optimal policy, or an optimal policy in general, are not guaranteed to exist [42]. Next, we show that an optimal stationary policy exists for Problem 4.2 and define the structure of the optimal policy.

Theorem 10. *There exists an optimal stationary policy for the CMDP in Problem 4.2 and it is randomized in at most a single point in the state-space $\mathcal{S}$.*

Proof. A sketch of the proof is given as follows: First, we show that Theorem 2.5, Proposition 3.2, and Lemma 3.9 of [43] hold for Problem 4.2 by showing that Assumptions 1-4 of [43] hold. Then, by Theorem 2.5 of [43], there exists an optimal stationary policy that is a mixture of two deterministic policies which differ in at most one state and there exists a randomization coefficient denoted by $p_\theta \in [0, 1]$ such that π^* satisfies the constraint with equality. The detailed proof can be obtained by following the same steps as in [40]. □

As a result of Theorem 10, we restrict our attention to the stationary policies in the rest of the paper.

4.2.1.1 Steady-State Analysis

In this section, we investigate steady-state behavior of the Markov Chain constructed by the states s_t as defined in Section 4.1, under a reasonable stationary policy. The MC has a unique steady state distribution if it is irreducible and positive recurrent [26, Ch. 6].

All states in the MC are reachable from the $\Delta = 1$ state, because for any $k \geq 1$; if the channel state is B between t and $t + k$, then age increases by k with probability

1. Hence, $\Pr(\Delta_{t+k} = k+1 \mid \Delta_t = 1) \geq P_B^k$. To show that the $\Delta = 1$ state is positive recurrent, we consider the expected time between consecutive transmissions. If the expected time is infinite, then the average energy cost would be 0 and such a policy would obviously be inferior to the ones that satisfy the constraint in (4.4) with equality. If the expected time between the transmissions is finite, then expected return time to $\Delta = 1$ state is finite and the MC is positive recurrent. Consequently, there are policies that lead to a steady-state distribution and we focus on such policies. We use $\Pr(\Delta = k)$ to denote the steady-state probability of the age being equal to k.

4.2.1.2 Structure of the Optimal Stationary Policy

In Problem 1, if there was no energy constraint ($\lambda \to \infty$), the transmitter would be able to take advantage of all transmission opportunities. Note that, if $\lambda \geq P_G$, such an unconstrained policy is feasible (due to the infinite battery assumption, the transmitter will never have to idle at a transmission opportunity.) Any policy that misses a transmission opportunity can only do worse, because in any sample path of the channel state process consisting of random realizations of G and B slots, the age graph of a policy that exploits all the G slots will be dominated by any other feasible age plot. That is, as the zero wait policy brings the age down to unity at all G slots, its age will be below or equal to that of any other feasible age graph attainable on the same sample path. Therefore, unlike in [36, 37], in the case that $\lambda \geq P_G$, the optimal policy is a zero-wait policy.

Having made this observation, for the rest of the paper, we focus on the nontrivial case $\lambda < P_G$. In the following, we show the optimality of a threshold policy that fully utilizes the energy constraint for remaining cases:

Theorem 11. *Let Θ be an integer for which there exists a stationary policy, π^*, such that*

(i) $\Pr(u = 1 \mid \Delta < \Theta, C = \mathrm{G}) = 0$

(ii) $\Pr(u = 1 \mid \Delta > \Theta, C = \mathrm{G}) = 1$

(iii) $\Pr(u = 1 \mid C = B) = 0$

(iv) $\Pr(u=1)=\lambda$

when π^* is employed. Then, π^* is optimal for Problem 1 such that for any stationary policy π;

$$\Delta^{\pi}(s_0) \geq \sum_{k=1}^{\infty} g(k)h(k) \tag{4.5}$$

where $h(k)$ is defined as:

$$h(k) = \begin{cases} \lambda, & k \leq \Theta \\ (1-\lambda\Theta)P_G P_B^{k-\Theta-1}, & k \geq \Theta+1 \end{cases} \tag{4.6}$$

Furthermore, (4.5) *is tight when $\pi \equiv \pi^*$.*

In this theorem, the optimal age penalty and the properties of the optimal policy are stated. The first two conditions on π^* suggest a threshold policy, while the latter conditions ensure that the policy does not waste energy.

Before laying out the proof of the theorem, we investigate the steady state distribution of the AoI. At any time t, the relation between $\Pr(\Delta_t = k)$ and $\Pr(\Delta_{t+1} = k+1)$ can be derived using (4.1):

$$\Pr(\Delta_{t+1} = k+1) = \Pr(\Delta_t = k)(1 - P_G \Pr(u_t = 1 \mid s_t = (k, G))) \tag{4.7}$$

As we restrict attention to stationary policies (without loss of optimality, by Thm. 1), the above equation can be rewritten as:

$$\frac{\Pr(\Delta = k+1)}{\Pr(\Delta = k)} = 1 - P_G \Pr(u = 1 \mid s = (k, G)) \tag{4.8}$$

From (4.8), $\Pr(\Delta = k+1) \leq \Pr(\Delta = k)$ and therefore the PMF of AoI at steady state is monotonic. The following Lemma uses the monotonicity to establish a lower bound on the age violation probability.

Lemma 6. *For any $\gamma \in \mathbb{Z}^+$,*

$$\Pr(\Delta \geq \gamma + 1) \geq 1 - \lambda\gamma \tag{4.9}$$

with equality if and only if

(i) $\Pr(u = 1 \mid \Delta < \gamma, C = \mathrm{G}) = 0$

(ii) $\Pr(u = 1 \mid C = B) = 0$

(iii) $\Pr(u = 1) = \lambda$

Proof. The state of $\Delta = 1$ corresponds one-to-one to the successful transmissions by the transmitter and the probability of a transmission occurring on average must not be greater than λ. Hence,

$$\Pr(\Delta = 1) \leq \Pr(u = 1) \leq \lambda \tag{4.10}$$

Equality in (4.10) holds iff a successful transmission happens with probability λ at steady state. In other words, the energy constraint shall be fully utilized and available energy shall not be wasted on transmitting while the channel is B, corresponding to the second and third properties. Due to the monotonicity, $\Pr(\Delta = k) \leq \lambda$ for any k as a result of (4.10). Finally,

$$\Pr(\Delta \geq \gamma + 1) = 1 - \sum_{k=1}^{\gamma} Pr(\Delta = k) \geq 1 - \lambda\gamma \tag{4.11}$$

Equality in (4.11) holds iff $\Pr(\Delta = k) = \lambda$ for all $k < \gamma$. In order for this to happen, there must be no successful transmission while the age is smaller than γ due to (4.8), yielding the first condition. □

Lemma 6 describes a lower limit on the age violation probabilities for small violation thresholds. If γ is larger than $1/\lambda$, then $1 - \lambda\gamma$ would be negative and the inequality in Lemma 6 would be loose. In order to support larger violation thresholds, we present Lemma 7.

Lemma 7. *For any $\gamma, m \in \mathbb{Z}^+$; if $\gamma > m$, then*

$$\Pr(\Delta \geq \gamma + 1) \geq P_B^{\gamma - m}(1 - \lambda m) \tag{4.12}$$

with equality if and only if

(i) $\Pr(u = 1 \mid \Delta < m, C = \mathrm{G}) = 0$

(ii) $\Pr(u = 1 \mid \Delta > m, C = \mathrm{G}) = 1$

(iii) $\Pr(u = 1 \mid C = B) = 0$

(iv) $\Pr(u = 1) = \lambda$

Proof. Let k be an arbitrary positive integer. If the channel state is B and AoI is k at time t, AoI at time $t + 1$ is $k + 1$ with probability 1. Therefore,

$$P_B \Pr(\Delta = k) \leq \Pr(\Delta = k + 1) \tag{4.13}$$

Through induction, we can show that for any $r \in \mathbb{Z}^+$,

$$P_B^r \Pr(\Delta = k) \leq \Pr(\Delta = k + r) \tag{4.14}$$

holds. Using this property, following relation between $\Pr(\Delta \geq k+1)$ and $\Pr(\Delta = k)$ is obtained:

$$\begin{aligned} \Pr(\Delta \geq & k + 1) = \sum_{r=1}^{\infty} \Pr(\Delta = k + r) \\ & \geq \sum_{r=1}^{\infty} P_B^r \Pr(\Delta = k) = \frac{P_B}{P_G} \Pr(\Delta = k) \end{aligned} \tag{4.15}$$

The fact that $\Pr(\Delta \geq k) - \Pr(\Delta \geq k + 1) = \Pr(\Delta = k)$ can be used to rewrite (4.15) as:

$$P_B \Pr(\Delta \geq k) \leq \Pr(\Delta \geq k + 1) \tag{4.16}$$

Through induction,

$$P_B^r \Pr(\Delta \geq k) \leq \Pr(\Delta \geq k + r) \tag{4.17}$$

follows for any $r \in \mathbb{Z}^+$. From Lemma 6,

$$\Pr(\Delta \geq m + 1) \geq 1 - \lambda m \tag{4.18}$$

For $k = m + 1$ and $r = \gamma - m$ in (4.17), we obtain:

$$\Pr(\Delta \geq \gamma + 1) \geq P_B^{\gamma - m} \Pr(\Delta \geq m + 1) \geq P_B^{\gamma - m}(1 - \lambda m) \tag{4.19}$$

Equality holds in (4.13) iff a transmission happens with probability 1 at $s = (k, G)$ state. Equality in (4.14)-(4.17) holds iff a transmission always takes place when the channel state is G and the AoI is greater than or equal to k. Due to the choice of $k = m+1$, *(ii)* is required for an equality. Rest of the equality conditions follow from (4.18) and Lemma 6. □

Finally, we prove Theorem 11 using Lemmas 6 and 7.

Proof of Theorem 11. Expected age penalty can be written in terms of the steady state probabilities of Δ:

$$
\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}E\left[\sum_{t=1}^{T} g(\Delta_t)\right] &= \sum_{k=1}^{\infty} g(k)\Pr(\Delta = k) \\
&= \sum_{k=1}^{\infty} g(k)(\Pr(\Delta \geq k) - \Pr(\Delta \geq k+1)) \\
&= g(1) + \sum_{k=1}^{\infty}(g(k+1) - g(k))\Pr(\Delta \geq k+1) \\
&\overset{(a)}{\geq} g(1) + \sum_{k=1}^{\Theta}(g(k+1) - g(k))(1-\lambda k) \\
&\quad + \sum_{k=\Theta+1}^{\infty}(g(k+1) - g(k))(1-\lambda\Theta)P_B^{k-\Theta} \\
&= \lambda\sum_{k=1}^{\Theta} g(k) + (1-\lambda\Theta)P_G\sum_{k=\Theta+1}^{\infty} g(k)P_B^{k-\Theta-1}
\end{aligned}
\tag{4.20}
$$

where (a) follows from Lemma 6 and Lemma 7. □

Corollary 3. *If the function g is strictly increasing, then equality in* (4.5) *holds if and only if the conditions (i)-(iv) are satisfied. In this case, π^* becomes the optimal stationary policy and π^* is unique.*

4.2.1.3 Derivation of the Threshold

In the previous section, we showed what the transmission probabilities should be under an optimal policy, except that we did not find the value of Θ. We also did not reveal the transmission probability of π^* when the age is equal to Θ. In this section, we fully derive the policy π^* that satisfies the conditions of Theorem 11. This policy is expressed in Theorem 12.

Theorem 12. *The optimal policy π^* for Problem 1 under IID channel states is a randomized threshold policy, which can be written as:*

$$
\pi^*(u_t = 1 \mid s_t = (\Delta_t, C_t)) = \begin{cases} 1, & \Delta_t > \Theta \textit{ and } C_t = G \\ p_\Theta, & \Delta_t = \Theta \textit{ and } C_t = G \\ 0, & \Delta_t < \Theta \textit{ or } C_t = B \end{cases}
\tag{4.21}
$$

where threshold Θ and randomization coefficient p_Θ are given as:

$$\Theta = \left\lfloor 1 + \frac{1}{\lambda} - \frac{1}{P_G} \right\rfloor \tag{4.22}$$

$$p_\Theta = \Theta - \left(\frac{1}{\lambda} - \frac{1}{P_G} \right) \tag{4.23}$$

Proof. In the proof, we shall derive the values of Θ and p_Θ. Note that there can be two different choices of thresholds depending on whether $p_\Theta \in [0, 1)$ or $p_\Theta \in (0, 1]$. We assume $p_\Theta \in (0, 1]$ without loss of generality to ensure that there exists a unique threshold and a unique set of parameters as a result of our analysis.

In Fig. 2.2, the state diagram for the policy above is illustrated. Corresponding state transition probabilities and total probability equation are as follows:

$$\Pr(\Delta = k) = \Pr(\Delta = 1) \text{ if } k \leq \Theta \tag{4.24}$$

$$\Pr(\Delta = \Theta + 1) = \Pr(\Delta = \Theta)(1 - p_\Theta P_G) \tag{4.25}$$

$$\Pr(\Delta = k) = \Pr(\Delta = k - 1)P_B \text{ if } k \geq \Theta + 2 \tag{4.26}$$

$$\sum_{k=1}^{\infty} \Pr(\Delta = k) = 1 \tag{4.27}$$

Note that the probability of making a succesful transmission and returning to the state $\Delta = 1$ is zero while the age is less than Θ, leading to (4.24). We obtain a closed-form solution of $\Pr(\Delta = k)$ by solving these equations together, with the first element of the series being equal to:

$$\Pr(\Delta = 1) = \frac{1}{\Theta - p_\Theta + \frac{1}{P_G}} \tag{4.28}$$

Due to the conditions *(iii)* and *(iv)* in Theorem 11, $\Pr(\Delta = 1) = \lambda$. Therefore,

$$\frac{1}{\Theta - p_\Theta + \frac{1}{P_G}} = \lambda \tag{4.29}$$

Finally, we use the fact that Θ is an integer and $p_\Theta \in (0, 1]$ to derive the unknown parameters as in (4.22) and (4.23). □

Corollary 4 (Optimal Age Violation Probability)**.** *If g function is set as $g(\Delta) = u(\Delta - \gamma)$ where $u(\cdot)$ is the unit step function such that $u(x) = \mathbb{1}_{x>0}$, expected age penalty would be equal to the age violation probability with a violation threshold γ. The optimal threhold policy π^* derived in Theorem 3 minimizes the age violation probability and optimal age violation probability can be computed in closed form as in the following:*

$$\Pr(\Delta > \gamma) \geq \begin{cases} 1 - \lambda\gamma, & \gamma \leq \Theta \\ P_B^{\gamma-\Theta}\left(1 - \lambda\Theta\right), & \gamma \geq \Theta. \end{cases} \tag{4.30}$$

Corollary 5 (Optimal Time Average AoI)**.** *The threshold policy π^* derived in Theorem 3 minimizes the time average AoI, when g function is set as the identity function such that $g(\Delta) = \Delta$. Then, optimal time average AoI can be computed in closed form as in the following:*

$$\lim_{T\to\infty} \frac{1}{T} E\left[\sum_{t=1}^{T} \Delta_t\right] \geq \frac{1}{P_G} + \lambda\Theta\left(\frac{1}{\lambda} - \frac{1}{P_G} - \frac{\Theta - 1}{2}\right). \tag{4.31}$$

4.2.2 Markovian Channel

In the Markovian channel case, the channel is characterized by the transition probabilities, p_{01}, p_{10}. For the special case of $p_{01} + p_{10} = 1$, the channel becomes memoryless with channel states becoming independent and identically distributed, which was thoroughly analyzed in the previous part. In the analysis of Markovian channel case, we propose a time-inhomogenous Markov chain that will investigate the state evolutions between successive transmissions. For reference, state transitions after m time slots are notated as $p_{ij}^{(m)}$:

$$\begin{aligned} \begin{bmatrix} p_{11}^{(m)} & p_{10}^{(m)} \\ p_{01}^{(m)} & p_{00}^{(m)} \end{bmatrix} &= \begin{bmatrix} p_{11} & p_{10} \\ p_{01} & p_{00} \end{bmatrix}^m \\ &= \frac{1}{p_{01} + p_{10}} \begin{bmatrix} p_{01} & p_{10} \\ p_{01} & p_{10} \end{bmatrix} + \frac{(1 - p_{10} - p_{01})^m}{p_{01} + p_{10}} \begin{bmatrix} p_{10} & -p_{10} \\ -p_{01} & p_{01} \end{bmatrix} \end{aligned} \tag{4.32}$$

Let $(Y_1, Y_2, \ldots)$ process be defined as the times between consecutive successful transmissions. This process is causal and $Y_1, Y_2, \ldots, Y_n$ has a joint probability measure that can be addressed. Consequently, we may refer to the marginal distribution of Y_n

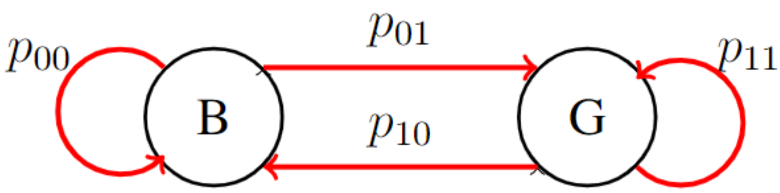


Figure 4.2: State Diagram of a Gilbert-Elliott Channel

in the rest of this paper. We can also transform the average energy constraint in (4.4) onto the Y_n process, because $\sum_{i=1}^{n} Y_i$ is the time of the n^{th} successful transmission:

$$\lim_{n\to\infty} E\left[\frac{\sum_{i=1}^{n} Y_i}{n}\right] = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} E[Y_i]}{n} = \frac{1}{\lambda} \tag{4.33}$$

We may also express the average age penalty in terms of Y_n process as follows:

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} g(\Delta_t) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n}\sum_{j=1}^{Y_i} g(j)}{\sum_{i=1}^{n} Y_i} \tag{4.34}$$

Overall, we may formulate the objective of this part as follows: Find a policy π such that, (4.34) is minimized while the energy constraint in (4.33) is satisfied.

The steady state probability of channel being in the G state is $P_G = \frac{p_{01}}{p_{01}+p_{10}}$. As is discussed earlier, if λ is greater than P_G, optimal policy is to transmit a packet whenever the channel is in G state. Hence, we shall focus solely on $\lambda < P_G$ case.

In the following, we present the main result of this section that asserts the optimality of a stationary threshold policy:

Theorem 13. *If $\lambda < P_G$, then the optimal policy π^* is a randomized threshold policy, which is described as:*

$$\Pr(a_t = 1 \mid \Delta_t, c_t) = \begin{cases} 1, & \Delta_t > \Theta \textit{ and } c_t = G \\ p_\Theta, & \Delta_t = \Theta \textit{ and } c_t = G \\ 0, & \Delta_t < \Theta \textit{ or } c_t = B \end{cases} \tag{4.35}$$

where the age threshold Θ is the greatest integer that satisfies the following inequality:

$$\Theta - ca^{\Theta} \le b \tag{4.36}$$

where

$$a = 1 - p_{10} - p_{01}$$

$$b = \frac{1}{\lambda} - \frac{1}{p_{01}} + \frac{1}{p_{01}+p_{10}}$$
$$c = \frac{1}{p_{01}} - \frac{1}{p_{01}+p_{10}}$$

and the randomization coefficient p_Θ is:

$$p_\Theta = \frac{b - \Theta + ca^\Theta}{1 + ca^\Theta - ca^{\Theta+1}} \tag{4.37}$$

In order to prove this theorem, we shall prove that a stationary policy is optimal and that the threshold policy described in Theorem 13 is optimal among the stationary policies. To this end, we will firstly show that given the expectation constraint on Y_i for an arbitrary i:

$$E[Y_i] = \sum_{k=1}^{\infty} \Pr(Y_i \geq k) = \gamma_i \tag{4.38}$$

the following total age penalty function between the i^{th} and $(i+1)^{\text{th}}$ transmissions is minimized if a threshold policy is deployed:

$$\tilde{g}_i \triangleq E\left[\sum_{j=1}^{Y_i} g(j)\right] = \sum_{k=1}^{\infty} \Pr(Y_i = k) \sum_{j=1}^{k} g(j) = \sum_{k=1}^{\infty} \Pr(Y_i \geq k) g(k) \tag{4.39}$$

After that, we shall show the stationarity of the optimal policy.

4.2.2.1 Optimality of a Threshold

In this section, we will investigate the system between two consecutive transmissions, namely i^{th} and $(i+1)^{\text{th}}$ successful transmissions. We denote the expected value of Y_i by γ_i, as in (4.38). Note that, we assume γ_i is a real number and that $E[Y_i] < \infty$. We comment on the special case of $E[Y_i] = \infty$ at the end of this section. With a slight abuse of notation, we use Y and γ instead of Y_i and γ_i, respectively.

We use the Markov chain illustrated in Fig. 4.2 to model the evolution of the system between i^{th} and $(i+1)^{\text{th}}$ successful transmissions. The three states of this Markov Chain are T (transmitted), G (good, no transmission yet), B (bad, no transmission yet). We add an additional state to the Markov Chain to the channel model illustrated in Fig. 4.3. This state T is an absorbing state and stands for the $(i+1)^{\text{th}}$ transmission in the network, that is effectively the end of this MC.

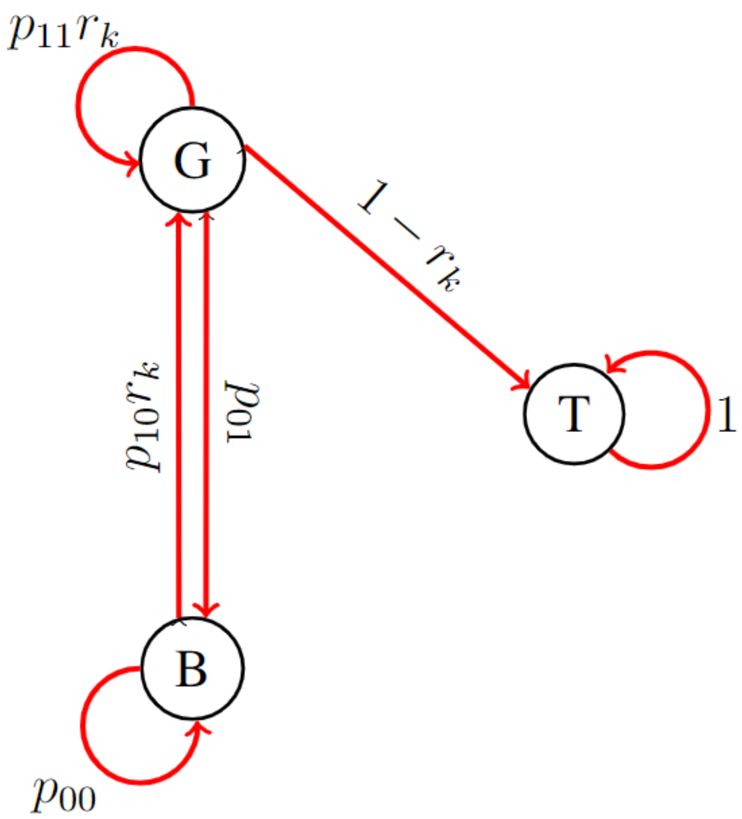


Figure 4.3: A time-inhomogenous Markov Chain to model the system between consecutive transmissions

The k^{th} iteration of this MC stands for the k^{th} time slot after a successful transmission, so k is equal to the AoI. We represent the probability of not making a transmission while the channel is in G state at k^{th} iteration as r_k. The value of r_k is determined according to the policy π and may be influenced by the previous transmission times. In this section, we analyze over an arbitrary sequence of r_k series such that (4.38) is satisfied and determine what should the r_k sequence be so that total age penalty is minimized. The sequence r_k does not necessarily take the same value for each k, so the MC in Fig. 4.3 should be considered as a time-inhomogenous Markov chain.

We introduce the variables x_k and y_k to denote the probabilities of MC being at state G and B at k^{th} iteration, respectively. A successful transmission may not occur while the channel is in B state, so the channel in the same slot as a successful transmission must be in G state. Therefore, the initial state is:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} p_{11} \\ p_{10} \end{bmatrix} \tag{4.40}$$

The pair of (x_k, y_k) evolves as:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} p_{11}r_k & p_{01} \\ p_{10}r_k & p_{00} \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} \tag{4.41}$$

If the MC is at state T at the k^{th} iteration, then a successful transmission must have happened before step k, i.e. $Y < k$. Converse of this holds as well. Hence, $\Pr(Y \geq k) = x_k + y_k$. Then the expectation condition in (4.38) can be used to establish the following relation between x_k, y_k series and γ.

$$\sum_{k=1}^{\infty}(x_k + y_k) = \sum_{k=1}^{\infty} \Pr(Y \geq k) = E[Y] = \gamma \tag{4.42}$$

Following proposition discloses some of the main characteristics of the x_k and y_k series.

Proposition 6. *i) If $k > l$, then*

$$y_k \geq y_l p_{00}^{k-l} \tag{4.43}$$

holds for $\forall k, l \in \mathbf{Z}^+$.

ii) For $\forall j, k \in \mathbf{Z}^+$,

$$\frac{\mathrm{d}x_k}{\mathrm{d}r_j} \geq 0, \quad \frac{\mathrm{d}y_k}{\mathrm{d}r_j} \geq 0 \tag{4.44}$$

holds.

Proof. See Appendix E. □

In the following two Lemmas, we establish a lower bound the sum of x_k and y_k series to establish some key properties on the distribution of Y; which will then be used to show the optimality of a threshold policy.

Lemma 8. *Series C_m is defined as:*

$$C_m = \left(\sum_{k=1}^{m} x_k + y_k\right) + y_m\left(1 - \frac{p_{00}}{p_{10}}\right) \tag{4.45}$$

Then,

(i) $\frac{\mathrm{d}C_m}{\mathrm{d}r_j} \geq 0$ *for* $\forall m, j \in \mathbf{Z}^+$.

(ii) $C_m \leq m + p_{10}^{(m)}\left(1 - \frac{p_{00}}{p_{10}}\right)$ *for* $\forall m \in \mathbf{Z}^+$.

Proof. Due to causality, the value of r_j has no effect on the system until the $(j+1)^{\text{th}}$ iteration and has no relevance to x_k and y_k if $k \leq j$. Therefore, $\frac{\mathrm{d}C_m}{\mathrm{d}r_j} = 0$ if $m \leq j$. If $m = j+1$, then (4.41) is used to calculate following derivations:

$$\frac{\mathrm{d}(x_{j+1} + y_{j+1})}{\mathrm{d}r_j} = x_j, \qquad \frac{\mathrm{d}y_{j+1}}{\mathrm{d}r_j} = x_j p_{10} \tag{4.46}$$

leading to:

$$\frac{\mathrm{d}C_{j+1}}{\mathrm{d}r_j} = x_j(p_{10} + p_{01}) \geq 0 \tag{4.47}$$

Next, we investigate the case of $m \geq j+2$. For any $n \geq 2$,

$$C_n - C_{n-1} = x_n + y_n + (y_n - y_{n-1})\left(1 - \frac{p_{00}}{p_{10}}\right) \tag{4.48}$$

We may show that $\frac{\mathrm{d}(C_n - C_{n-1})}{\mathrm{d}r_j} \geq 0$ for any $n \geq j+2$ to prove the first property. From (4.41):

$$\frac{\mathrm{d}y_n}{\mathrm{d}r_j} = p_{10} r_{n-1} \frac{\mathrm{d}x_{n-1}}{\mathrm{d}r_j} + p_{00} \frac{\mathrm{d}y_{n-1}}{\mathrm{d}r_j} \quad \text{for any } n \geq j+2 \tag{4.49}$$

$$\frac{\mathrm{d}(x_n + y_n)}{\mathrm{d}r_j} = r_{n-1} \frac{\mathrm{d}x_{n-1}}{\mathrm{d}r_j} + \frac{\mathrm{d}y_{n-1}}{\mathrm{d}r_j} \quad \text{for any } n \geq j+2 \tag{4.50}$$

We derive the expression in (4.48) to show monotonicity of $\frac{\mathrm{d}C_n}{\mathrm{d}r_j}$ for n:

$$\begin{aligned} \frac{\mathrm{d}C_n}{\mathrm{d}r_j} - \frac{\mathrm{d}C_{n-1}}{dr_j} &= \frac{\mathrm{d}(x_n + y_n)}{\mathrm{d}r_j} + \frac{\mathrm{d}(y_n - y_{n-1})}{\mathrm{d}r_j}\left(1 - \frac{p_{00}}{p_{10}}\right) \\ &= (p_{01} + p_{10}) r_{n-1} \frac{\mathrm{d}x_{n-1}}{\mathrm{d}r_j} + \frac{p_{00}p_{01}}{p_{10}} \frac{\mathrm{d}y_{n-1}}{\mathrm{d}r_j} \geq 0 \end{aligned} \tag{4.51}$$

Finally, (4.47) and (4.51) is used to conclude the proof of the first property:

$$\frac{\mathrm{d}C_m}{\mathrm{d}r_j} = \frac{\mathrm{d}C_{j+1}}{\mathrm{d}r_j} + \sum_{n=j+2}^{m} \left(\frac{\mathrm{d}C_n}{\mathrm{d}r_j} - \frac{\mathrm{d}C_{n-1}}{\mathrm{d}r_j}\right) \geq 0 \tag{4.52}$$

As a result of the first property, C_m is maximized when $r_1 = r_2 = \ldots = r_{m-1} = 1$. In this case, $x_n + y_n = 1$ for all $1 \leq n \leq m$ and $y_m = p_{10}^{(m)}$; second property follows. □

Lemma 9. *Let* $D_m \triangleq m + p_{10}^{(m)}\left(1 - \frac{p_{00}}{p_{10}}\right)$. *For* $\forall m, n \in \mathbf{Z}^+$*; if* $n \geq 2$, *then*

$$\sum_{k=m+n}^{\infty} (x_k + y_k) \geq \frac{p_{10} p_{00}^{n-2}}{p_{10} + p_{01}} (\gamma - D_m) \tag{4.53}$$

Proof. Say we fix the values of x_{m+n-1} and y_{m+n-1}. Then, the sum $\sum_{k=m+n}^{\infty}(x_k+y_k)$ would be minimized if $r_j = 0$ for all $j \geq m+n-1$, due to Prop 6. Hence,

$$\sum_{k=m+n}^{\infty} (x_k + y_k) \geq \sum_{k=m+n}^{\infty} y_{m+n-1} p_{00}^{k-m-n} = \frac{y_{m+n-1}}{p_{01}} \tag{4.54}$$

Using Prop. 6, above statement can be written in a more general form that holds for any $n-1 \geq d \geq 1$:

$$\sum_{k=m+n}^{\infty} (x_k + y_k) \geq y_{m+d} \frac{p_{00}^{n-d-1}}{p_{01}} \tag{4.55}$$

Next, suppose (4.53), the lemma statement, is not true:

$$\sum_{k=m+n}^{\infty} (x_k + y_k) < \frac{p_{10} p_{00}^{n-2}}{p_{10} + p_{01}} (\gamma - D_m) \tag{4.56}$$

From (4.55) and (4.56), we would have the following inequality:

$$y_{m+d} < \frac{p_{01} p_{10} p_{00}^{d-1}}{p_{10} + p_{01}} (\gamma - D_m) \tag{4.57}$$

Transition probabilities in (4.41) can bu used to derive that:

$$x_{k+1} + y_{k+1} = y_k \left(1 - \frac{p_{00}}{p_{10}}\right) + \frac{y_{k+1}}{p_{10}} \tag{4.58}$$

The equation in (4.58) is used to derive:

$$\begin{aligned}
\sum_{k=m+1}^{m+n-1} x_k + y_k &= y_m \left(1 - \frac{p_{00}}{p_{10}}\right) + \left(\sum_{k=m+1}^{m+n-2} y_k\right) \left(\frac{p_{01} + p_{10}}{p_{10}}\right) + \frac{y_{m+n-1}}{p_{10}} \\
&\overset{(a)}{<} y_m \left(1 - \frac{p_{00}}{p_{10}}\right) + \sum_{d=1}^{n-2} p_{01} p_{00}^{d-1} (\gamma - D_m) + \frac{p_{01} p_{00}^{n-2}}{p_{10} + p_{01}} (\gamma - D_m) \\
&= y_m \left(1 - \frac{p_{00}}{p_{10}}\right) + \left(1 - \frac{p_{10} p_{00}^{n-2}}{p_{10} + p_{01}}\right) (\gamma - D_m)
\end{aligned} \tag{4.59}$$

where (a) follows from (4.57). In Lemma 8, we had shown that $C_m \leq D_m$ for all $m \in \mathbb{Z}^+$. We use this inequality in the following:

$$\begin{aligned}
\sum_{k=1}^{m+n-1} x_k + y_k &< \sum_{k=1}^{m} (x_k + y_k) + y_m \left(1 - \frac{p_{00}}{p_{10}}\right) + \left(1 - \frac{p_{10} p_{00}^{n-2}}{p_{10} + p_{01}}\right) (\gamma - D_m) \\
&= C_m + \left(1 - \frac{p_{10} p_{00}^{n-2}}{p_{10} + p_{01}}\right) (\gamma - D_m) \\
&\leq D_m + \left(1 - \frac{p_{10} p_{00}^{n-2}}{p_{10} + p_{01}}\right) (\gamma - D_m)
\end{aligned} \tag{4.60}$$

Finally, we get the following contradiction, where (a) follows from (4.56) and (4.61):

$$\gamma = \sum_{k=1}^{\infty}(x_k + y_k) = \sum_{k=1}^{m+n-1}(x_k + y_k) + \sum_{k=m+n}^{\infty}(x_k + y_k)$$
$$< D_m + \left(1 - \frac{p_{10}p_{00}^{n-2}}{p_{10}+p_{01}}\right)(\gamma - D_m) + \frac{p_{10}p_{00}^{n-2}}{p_{10}+p_{01}}(\gamma - D_m) = \gamma \tag{4.61}$$

Due to the contradiction, (4.56) must be untrue and therefore (4.53) holds. □

In Lemma 2, the equality may be obtained setting $r_j = 1$ for all $j < m$ and $r_j = 0$ for all $j > m$. As such, we argue that an age threshold may be used to minimize. We use this as the basic intuition behind the threshold policy and define the following $\pi^{(T)}$ sequence.

Definition 2. *Let the* ***age threshold****, Θ, be the greatest integer such that*

$$\Theta + \frac{p_{10}^{(\Theta)}}{p_{01}} \leq \gamma \tag{4.62}$$

or equivalently,

$$\Theta - ca^{\Theta} \leq b \tag{4.63}$$

where

$$a = 1 - p_{10} - p_{01}$$
$$b = \gamma - c$$
$$c = \frac{1}{p_{01}} - \frac{1}{p_{01}+p_{10}}$$

Threshold policy *$\pi^{(T)}$ is defined as follows: The probability of not making a transmission while the channel is G, after m time slots have passed since the last transmission is:*

$$r_m^{(T)} = \begin{cases} 1, & m < \Theta \\ \frac{b-\Theta+ca^{\Theta}}{1+ca^{\Theta}-ca^{\Theta+1}}, & m = \Theta \\ 0, & m > \Theta \end{cases} \tag{4.64}$$

Analogous to Y, let $Y^{(T)}$ be defined as the time between the consecutive transmissions if π^* was employed. Then,

$$\Pr(Y^{(T)} \geq m) = \begin{cases} 1, & m \leq \Theta \\ \frac{p_{01}}{p_{10}+p_{01}}\gamma + \frac{p_{10}}{p_{10}+p_{01}}D_{\Theta} - \Theta, & m = \Theta + 1 \\ \frac{p_{01}p_{10}p_{00}^{m-\Theta-2}}{p_{10}+p_{01}}\left(\gamma - D_{\Theta}\right), & m \geq \Theta + 2 \end{cases} \tag{4.65}$$

Lemma 10. *For any $m \in \mathbf{Z}^+$,*

$$\sum_{k=m}^{\infty} \Pr(Y \geq k) \geq \sum_{k=m}^{\infty} \Pr(Y^{(T)} \geq k) \tag{4.66}$$

Proof. Using (4.65),

$$\sum_{k=m}^{\infty} \Pr(Y^{(T)} \geq k) = \begin{cases} \gamma - m + 1, & m \leq \Theta + 1 \\ \frac{p_{10} p_{00}^{m-\Theta-2}}{p_{10}+p_{01}} \left(\gamma - D_\Theta\right), & m \geq \Theta + 2 \end{cases} \tag{4.67}$$

If $m \geq \Theta + 2$, the inequality follows directly from Lemma 2. If $m \leq \Theta + 1$, then

$$\begin{aligned} \sum_{k=m}^{\infty} \Pr(Y \geq k) &= \gamma - \sum_{k=1}^{m-1} \Pr(Y \geq k) \\ &\geq \gamma - (m-1) = \sum_{k=m}^{\infty} \Pr(Y^{(T)} \geq k) \end{aligned} \tag{4.68}$$

□

Finally, we use Lemma 3 to show that the threshold policy achieves optimal results over the minimization of age penalty.

Theorem 14. *The threshold policy minimizes the expected age penalty given the $E[Y] = \gamma$ constraint:*

$$\tilde{g} \geq \tilde{g}^{(T)} \tag{4.69}$$

where $\tilde{g}$ and $\tilde{g}^{(T)}$ are defined as:

$$\tilde{g} \triangleq E\left[\sum_{j=1}^{Y} g(j)\right] = \sum_{k=1}^{\infty} \Pr(Y = k) \sum_{j=1}^{k} g(j) \tag{4.70}$$

$$\tilde{g}^{(T)} \triangleq E\left[\sum_{j=1}^{Y^{(T)}} g(j)\right] = \sum_{k=1}^{\infty} \Pr(Y^{(T)} = k) \sum_{j=1}^{k} g(j) \tag{4.71}$$

Proof. Define A_m and B_m series to be:

$$A_m = \sum_{k=m}^{\infty} \Pr(Y \geq k), \quad B_m = \sum_{k=m}^{\infty} \Pr(Y^{(T)} \geq k) \tag{4.72}$$

Then, $A_m \geq B_m$ for all $m \in \mathbf{Z}^+$ due to Lemma 3 and:

$$\begin{aligned}
\sum_{k=1}^{\infty}(\Pr(Y \geq k) - \Pr(Y^{(T)} \geq k))g(k) &= \sum_{k=1}^{\infty}(A_k - A_{k+1} - B_k + B_{k+1})g(k) \\
&= \sum_{k=1}^{\infty}(A_k - B_k)g(k) - \sum_{k=1}^{\infty}(A_{k+1} - B_{k+1})g(k) \\
&= \sum_{k=1}^{\infty}(A_k - B_k)g(k) - \sum_{k=2}^{\infty}(A_k - B_k)g(k-1) \\
&= (A_1 - B_1)g(1) + \sum_{k=2}^{\infty}(A_k - B_k)(g(k) - g(k-1)) \overset{(a)}{\geq} 0
\end{aligned} \tag{4.73}$$

where (a) follows from $A_k \geq B_k$ and $g(k) \geq g(k-1)$ for all k. □

4.2.2.2 Optimality of Stationarity

In the previous subsection, we have shown the optimization process of Y given $E[Y] = \gamma$. Let $f(\gamma)$ be the minimum achievable age penalty if $E[Y] = \gamma$.

$$\begin{aligned}
f(\gamma) &= \sum_{k=1}^{\infty} \Pr(Y^{(T)} \geq k)g(k) \\
&= \sum_{k=1}^{\Theta} g(k) + \left(\frac{p_{01}}{p_{10}+p_{01}}\gamma + \frac{p_{10}}{p_{10}+p_{01}}D_{\Theta} - \Theta\right) g(\Theta+1) \\
&+ \sum_{k=\Theta+2}^{\infty} \frac{p_{01}p_{10}p_{00}^{k-\Theta-2}}{p_{10}+p_{01}} (\gamma - D_{\Theta})\, g(k)
\end{aligned} \tag{4.74}$$

Derivative of f is:

$$\frac{\mathrm{d}f}{\mathrm{d}\gamma} = \frac{p_{01}}{p_{10}+p_{01}} g(\Theta+1) + \sum_{k=\Theta+2}^{\infty} \frac{p_{01}p_{10}p_{00}^{k-\Theta-2}}{p_{10}+p_{01}} g(k) \tag{4.75}$$

Let γ_1 and γ_2 be two real numbers such that $\gamma_2 \geq \gamma_1$. Let Θ_1 and Θ_2 be the age thresholds when $E[Y]$ is γ_1 and γ_2, respectively. Then, $\Theta_2 \geq \Theta_1$ and

$$\left.\frac{\mathrm{d}f}{\mathrm{d}\gamma}\right|_{\gamma=\gamma_2} \geq \left.\frac{\mathrm{d}f}{\mathrm{d}\gamma}\right|_{\gamma=\gamma_1} \tag{4.76}$$

since $g(k)$ is non-decreasing. Consequently, the derivative of f is non-decreasing, hence, f is convex. Now let's return the original $(Y_1, Y_2, \ldots)$ process. Due to the mean energy constraint:

$$\begin{aligned}\lim_{n\to\infty} E\left[\frac{\sum_{i=1}^{n} Y_i}{n}\right] &= \lim_{n\to\infty} \frac{\sum_{i=1}^{n} E[Y_i]}{n} \\ &= \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \gamma_i}{n} = E[\gamma] = \frac{1}{\lambda}\end{aligned} \tag{4.77}$$

Mean age penalty is:

$$\begin{aligned}\lim_{n\to\infty} E\left[\frac{\sum_{i=1}^{n}\sum_{j=1}^{Y_i} g(j)}{\sum_{i=1}^{n} Y_i}\right] &\geq \lambda \lim_{n\to\infty} E\left[\frac{\sum_{i=1}^{n}\sum_{j=1}^{Y_i} g(j)}{n}\right] \\ &\overset{(a)}{\geq} \lambda \lim_{n\to\infty} E\left[\frac{\sum_{i=1}^{n} f(\gamma_i)}{n}\right] \\ &= \lambda E[f(\gamma)]\end{aligned} \tag{4.78}$$

where (a) follows from Theorem 1. Finally, from Jensen's Inequality,

$$E[f(\gamma)] \geq f(E[\gamma]) = f(1/\lambda) \tag{4.79}$$

Equality occurs if $E[Y_i] = 1/\lambda$ for all $i \in \mathbf{Z}^{+}$. As a result, mean age penalty is minimized if all instances of Y_i are expected to use the same amount of energy and a threshold policy is employed for each of them.

4.3 Numerical Results and Discussion

We compare the performance of our threshold policy to a uniform transmission policy that performs a transmission at any time with probability $\frac{\lambda}{P_G}$ while the channel state is G. The value of $\frac{\lambda}{P_G}$ is chosen such that the time average energy constraint is fully utilized by both policies and a fair comparison can be made, however, two policies converge to a zero-wait policy as λ approaches P_G with diminishing differences in terms of performance. Note that, we assume λ to be smaller than P_G, as explained in Section 4.2.1.2.

We run Monte Carlo simulations for 10^6 time slots with 100 iterations and for $P_G = 0.5$. Fig. 4.4 depicts the age violation probability of the optimal threshold policy and uniform transmission policy for a violation threshold of 15 time slots. We observe that the age violation probability is reduced substantially compared to the uniform policy, especially when λ is between 0 and P_G and far from both extremes. The results were

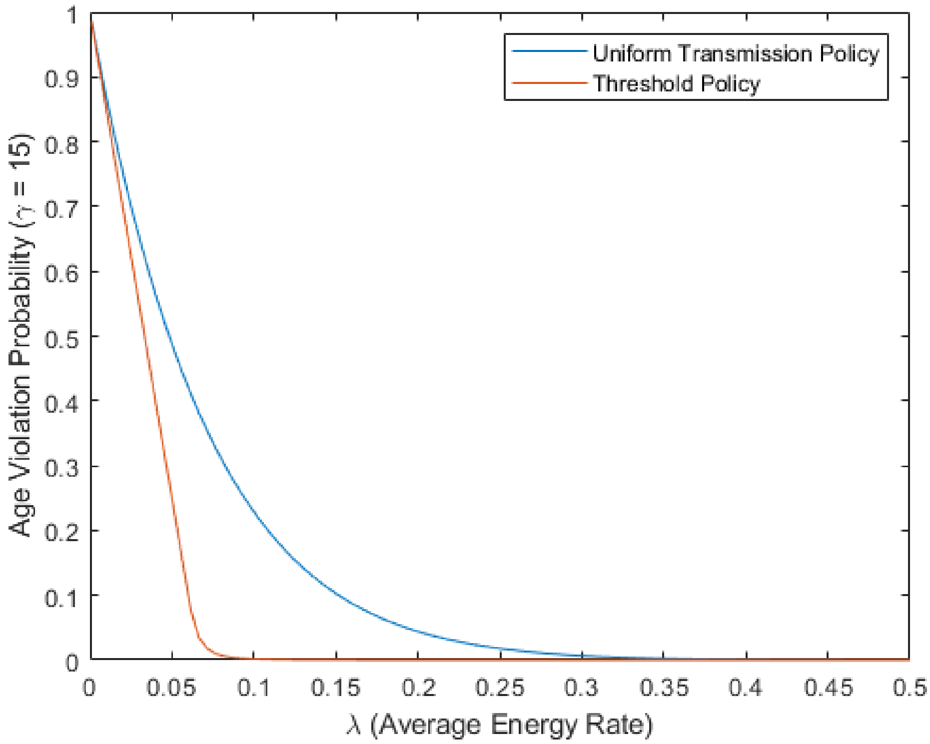


Figure 4.4: Age violation probability vs average energy rate

verified to be consistent with the theoretical findings. In Fig. 4.5, the optimal time average age penalty is illustrated for linear ($g(\Delta) = \Delta$) and exponential ($g(\Delta) = 1.5^{\Delta-1}$) age penalty functions. We observe that optimal average age penalty changes linearly when the threshold Θ stays fixed within a limited range of λ, however, the plots resemble a geometrical decay over a long range of λ in which Θ changes with λ, as in Fig. 4.4.

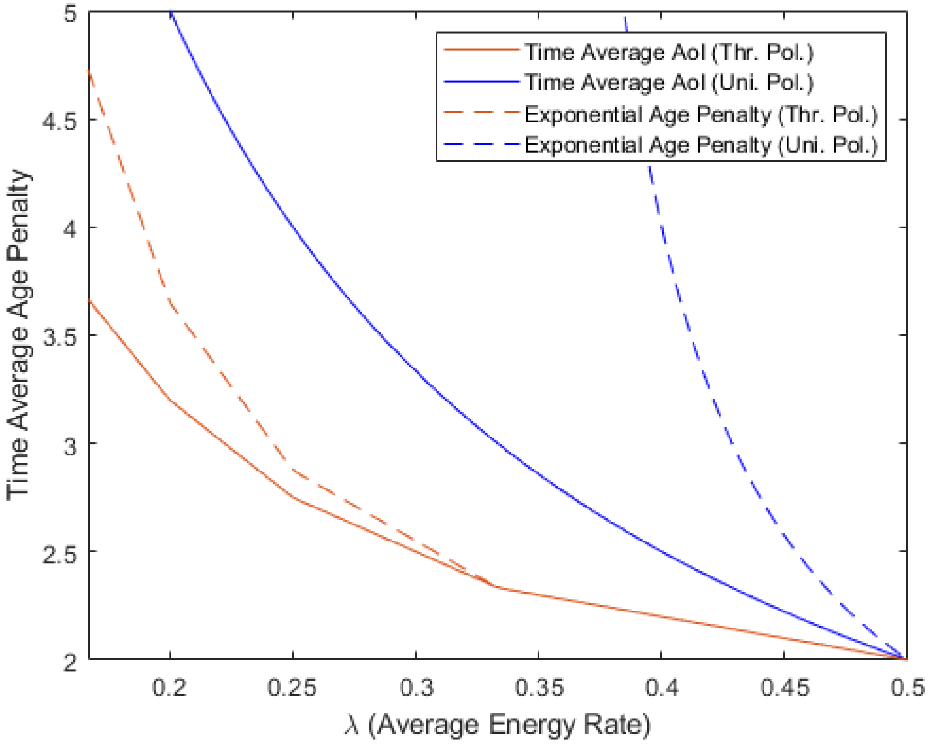


Figure 4.5: Time average age penalty under different penalty functions

CHAPTER 5

CONCLUSIONS

In this thesis, the information freshness and throughput performance of our recent random access and wireless energy transfer schemes were studied.

In Chapter 2, we have presented a comprehensive steady-state analysis of *threshold-ALOHA*, which is an age-aware modification of slotted ALOHA proposed in [12]. In threshold-ALOHA each terminal suspends its transmissions until its age exceeds a certain threshold, and once age exceeds the threshold, it attempts transmission with constant probability τ, just as in standard slotted ALOHA. We have analyzed time-average expected age attained, and explored its scaling with network size. We adopted the *generate-at-will* model where each time a user attempts transmission, it generates a fresh packet, accordingly every time a successful transmission occurs, the age of the corresponding flow is reset to 1. We have firstly derived the steady state solutions of DTMC that was formed in [12] and subsequently found the distribution of number of active users. We have shown that the policy converges to running slotted ALOHA with fewer sources: on average about one fifth of the users is active at any time. We then formulated an expression for avg. AoI and derived optimal parameters of the policy. This resolved the conjectures in [12] by confirming that the optimal age threshold and transmission probability are $2.2n$ and $4.69/n$, respectively. We have found optimal avg. AoI to be $1.4169n$, which is half of what is achievable using slotted ALOHA while the loss from the maximum achievable throughput of e^{-1} is below 1%.

In Chapter 3, we introduced a novel random access policy called MuMiSTA, a reservation based extension of the threshold Aloha policy. We have derived the maximum achievable throughput for any number of mini slots. We have constructed an analogy between threshold Aloha and MuMiSTA and found average AoI expressions for MuMiSTA. It was shown that MuMiSTA can achieve average AoI that is very close to the ideal round-robin case while keeping its random-access structure.

In Chapter 4, we designed a point-to-point information retrieval policy that minimizes a generalized age penalty, on a binary ON/OFF channel with a power constrained information pulling receiver. Modeling the problem as a CMDP, we showed that there exists a threshold policy that is optimal for the problem. We computed the threshold. The optimal time average Age of Information and age violation probabilities were found as corollaries to our main findings. We also unveiled an optimal policy for temporally correlated channels. Finally, we illustrated the performance impact of using this age-optimized policy, by comparing it to a benchmark uniform policy with the same energy expenditure.

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APPENDIX A

PROOF OF LEMMA 2

We firstly prove that properties of Lemma hold for $s = 1, 2, \ldots, \Gamma - 1$. Property (i) and (ii) follows from Prop. 3 (i), $\pi(S_1^{\mathbf{P}}) = \pi_{m_1}$ and $\pi(S_2^{\mathbf{P}}) = \pi_{m_2}$. Property (iii) follows from the same property, albeit not directly:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\pi(S_1^{\mathbf{P}})}{n\,\pi(S_2^{\mathbf{P}})} &= \lim_{n\to\infty} \frac{\pi_{m_1}}{n\pi_{m_1-1}} \\
&\overset{(a)}{=} \lim_{n\to\infty} \frac{1-(m_1-1)\tau(1-\tau)^{m_1-2}}{n\tau(1-\tau)^{m_1-1}} \\
&= \frac{e^{k\alpha}}{\alpha} - k
\end{aligned} \tag{A.1}$$

where (a) follows from (2.12).

Next, we calculate the steady state probabilities of the states in $\mathbf{P}^{\Gamma}$ where $s = \Gamma$. We firstly show that $\pi(S_1^{\mathbf{P}}) = \pi_{m_1}$. Assuming that the current state is $S_1^{\mathbf{P}}$, if $1 \notin \{u_1, u_2, \ldots, u_{n-m_1-1}\}$, then previous state must be of one of the following types:

- $(\Gamma - 1, m_1, \{u_1 - 1, u_2 - 1, \ldots, u_{n-m_1-1} - 1\})$
- $(\Gamma - 1, m_1 - 1, \{\Gamma - 1, u_1 - 1, u_2 - 1, \ldots, u_{n-m_1-1} - 1\})$

Steady state probability expression for states of these types are given in Prop. 3 (ii). Steady state probabilities for states of the first type and second type are π_{m_1} and π_{m_1-1}, respectively. Steady state probability of $S_1^{\mathbf{P}}$ can be derived using the steady state probabilities of preceding states along with their transition probabilities:

$$\pi(S_1^{\mathbf{P}}) = \pi_{m_1}(1 - m_1\tau(1-\tau)^{m_1-1}) + \pi_{m_1-1}m_1(1-(m_1-1)\tau(1-\tau)^{m_1-2}) = \pi_{m_1} \tag{A.2}$$

Resulting π_{m_1} is obtained through the ratio given in (2.12). Now, we calculate the steady state probability for the case $1 \in \{u_1, u_2, \ldots, u_{n-m_1-1}\}$, following similar steps. W.l.o.g., assume that $u_{n-m_1-1} = 1$. Then previous state must be one of the following types:

- $(\Gamma - 1, m_1 + 1, \{u_1 - 1, u_2 - 1, \ldots, u_{n-m_1-2} - 1\})$
- $(\Gamma - 1, m_1, \{\Gamma - 1, u_1 - 1, u_2 - 1, \ldots, u_{n-m_1-2} - 1\})$

Steady state probabilities for states of the first type and second type are π_{m_1+1} and π_{m_1}, respectively. Steady state probability of $S_1^{\mathbf{P}}$ is derived as:

$$\pi(S_1^{\mathbf{P}}) = \pi_{m_1+1}\tau(1-\tau)^{m_1} + \pi_{m_1}(m_1)\tau(1-\tau)^{m_1-1} = \pi_{m_1} \tag{A.3}$$

Due to symmetry, $\pi(S_2^{\mathbf{P}}) = \pi_{m_2}$. Property (i) and (ii) follows from Prop. 3 (i) and Property (iii) follows from (A.1).

Finally, we prove that properties of the Lemma hold for $\forall s \geq \Gamma$ by induction. Initial case $s = \Gamma$ has been covered above. We assume $s > \Gamma$ and that above properties hold for all states of $\mathbf{P}^{\Gamma}$ in which age of the pivot source is smaller than s. Then we prove property (i) in two separate cases:

Case 1. If $1 \notin \{u_1, u_2, \ldots, u_{n-m-1}\}$

In order to make the equations easier to read, we shorten steady state probability expressions in the following way:

$$\pi_m^{(s)} = \pi(s, m, \{u_1, u_2, \ldots, u_{n-m-1}\}) = \pi(S_1^{\mathbf{P}}) \tag{A.4}$$
$$\pi_m^{(s-1)} = \pi(s-1, m, \{u_1 - 1, u_2 - 1, \ldots, u_{n-m-1} - 1\}) = \pi(Q_1^{\mathbf{P}}) \tag{A.5}$$
$$\pi_{m-1}^{(s-1)} = \pi(s-1, m-1, \{\Gamma - 1, u_1 - 1, u_2 - 1, \ldots, u_{n-m-1} - 1\}) \tag{A.6}$$

Steady state prob. of the states that can precede a state of $(s, m, \{u_1, u_2, \ldots, u_{n-m-1}\})$ are $\pi_m^{(s-1)}$ or $\pi_{m-1}^{(s-1)}$. Value of $\pi_m^{(s)}$ is calculated as:

$$\pi_m^{(s)} = \pi_m^{(s-1)}(1 - (m+1)\tau(1-\tau)^m) + \pi_{m-1}^{(s-1)}(m+1)(1 - m\tau(1-\tau)^{m-1}) \tag{A.7}$$

Then,

$$\begin{aligned}\lim_{n\to\infty}\frac{\pi_m^{(s)}}{\pi_m^{(s-1)}} &= \lim_{n\to\infty}\frac{\pi_m^{(s-1)}(1-(m+1)\tau(1-\tau)^m)+\pi_{m-1}^{(s-1)}(m+1)(1-m\tau(1-\tau)^{m-1})}{\pi_m^{(s-1)}}\\ &= \lim_{n\to\infty} 1-(m+1)\tau(1-\tau)^m+\frac{\pi_{m-1}^{(s-1)}}{\pi_m^{(s-1)}}(m+1)(1-m\tau(1-\tau)^{m-1})\\ &= \lim_{n\to\infty} 1-\frac{m+1}{n}(n\tau)(1-\tau)^m+\frac{n\,\pi_{m-1}^{(s-1)}}{\pi_m^{(s-1)}}\frac{m+1}{n}(1-\frac{m}{n}(n\tau)(1-\tau)^{m-1})\\ &\overset{(a)}{=} \lim_{n\to\infty} 1-k\alpha e^{-k\alpha}+\frac{1}{\frac{e^{k\alpha}}{\alpha}-k}k(1-k\alpha e^{-k\alpha})=1\end{aligned} \tag{A.8}$$

where (a) follows from property (iii).

Case 2. If $1\in\{u_1,u_2,\ldots,u_{n-m-1}\}$, then w.l.o.g. $u_{n-m-1}=1$

In order to make the equations easier to read, we shorten steady state probability expressions in the following way:

$$\pi_m^{(s)} = \pi(s,m,\{u_1,u_2,\ldots,u_{n-m-2},1\}) = \pi(S_1^{\mathbf{P}}) \tag{A.9}$$

$$\pi_m^{(s-1)} = \pi(s-1,m,\{\Gamma-1,u_1-1,u_2-1,\ldots,u_{n-m-2}-1\}) = \pi(Q_1^{\mathbf{P}}) \tag{A.10}$$

$$\pi_{m+1}^{(s-1)} = \pi(s-1,m+1,\{u_1-1,u_2-1,\ldots,u_{n-m-2}-1\}) \tag{A.11}$$

Steady state prob. of the states that preceding a state of $(s,m,\{u_1,u_2,\ldots,u_{n-m-2},1\})$ are $\pi_m^{(s-1)}$ or $\pi_{m+1}^{(s-1)}$. Value of $\pi_m^{(s)}$ is calculated as:

$$\pi_m^{(s)} = \pi_{m+1}^{(s-1)}\tau(1-\tau)^m+\pi_m^{(s-1)}m\tau(1-\tau)^{m-1} \tag{A.12}$$

Then,

$$\begin{aligned}\lim_{n\to\infty}\frac{\pi_m^{(s)}}{\pi_m^{(s-1)}} &= \lim_{n\to\infty}\frac{\pi_{m+1}^{(s-1)}\tau(1-\tau)^m+\pi_m^{(s-1)}m\tau(1-\tau)^{m-1}}{\pi_m^{(s-1)}}\\ &= \lim_{n\to\infty}\frac{\pi_{m+1}^{(s-1)}}{\pi_m^{(s-1)}}\tau(1-\tau)^m+m\tau(1-\tau)^{m-1}\\ &= \lim_{n\to\infty}\frac{\pi_{m+1}^{(s-1)}}{n\,\pi_m^{(s-1)}}(n\tau)(1-\tau)^m+\frac{m}{n}(n\tau)(1-\tau)^{m-1}\\ &= \lim_{n\to\infty}(\frac{e^{k\alpha}}{\alpha}-k)\alpha e^{-k\alpha}+k\alpha e^{-k\alpha}=1\end{aligned} \tag{A.13}$$

Thus, the proof of property (i) is completed. Next, for the case $m_1=m_2$,

$$\begin{aligned}\lim_{n\to\infty}\frac{\pi(S_1^{\mathbf{P}})}{\pi(S_2^{\mathbf{P}})} &= \lim_{n\to\infty}\frac{\pi(S_1^{\mathbf{P}})}{\pi(Q_1^{\mathbf{P}})}\frac{\pi(Q_2^{\mathbf{P}})}{\pi(S_2^{\mathbf{P}})}\frac{\pi(Q_1^{\mathbf{P}})}{\pi(Q_2^{\mathbf{P}})}\\ &\overset{(a)}{=} \lim_{n\to\infty}\frac{\pi(Q_1^{\mathbf{P}})}{\pi(Q_2^{\mathbf{P}})} \overset{(b)}{=} 1\end{aligned} \tag{A.14}$$

where (a) follows from property (i) and (b) follows from property (ii) since state of the pivot source for states $Q_1^{\mathbf{P}}$ and $Q_2^{\mathbf{P}}$ is $s-1$ and number of active sources is m_1 and m_2 respectively.

Similarly, for the case $m_1 = m_2 + 1$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\pi(S_1^{\mathbf{P}})}{\pi(n\, S_2^{\mathbf{P}})} &= \lim_{n\to\infty} \frac{\pi(S_1^{\mathbf{P}})}{\pi(Q_1^{\mathbf{P}})} \frac{\pi(Q_2^{\mathbf{P}})}{\pi(S_2^{\mathbf{P}})} \frac{\pi(Q_1^{\mathbf{P}})}{n\, \pi(Q_2^{\mathbf{P}})} \\
&\overset{(a)}{=} \lim_{n\to\infty} \frac{\pi(Q_1^{\mathbf{P}})}{n\, \pi(Q_2^{\mathbf{P}})} \\
&\overset{(b)}{=} \frac{e^{k\alpha}}{\alpha} - k
\end{aligned} \tag{A.15}$$

where (a) follows from property (i), (b) follows from property (iii) since state of the pivot source for states $Q_1^{\mathbf{P}}$ and $Q_2^{\mathbf{P}}$ is $s-1$ and number of active sources is m_1 and m_2 respectively.

APPENDIX B

PROOF OF THEOREM 2

We shall prove the following Lemma, from which Theorem 2 will follow as a special case for $(a, b) = (0, 1)$.

Lemma 11. *For $(a, b) \subseteq (0, 1)$, let k_0 be the only root of $f(k)$ in the interval (a, b) and $f'(k_0) < 0$, $lim_{k \to a} f(k) \neq 0$, $lim_{k \to b} f(k) \neq 0$. Then for the sequence $\epsilon_n = cn^{-1/3}$ where $c \in \mathbb{R}^+$,*

i)

$$\frac{\Pr\left(\left|\frac{m}{n} - k_0\right| \geq \epsilon_n, \frac{m}{n} \in (a, b)\right)\right)}{P_{nk_0}} \to 0 \tag{B.1}$$

ii)

$$\Pr\left(\left|\frac{m}{n} - k_0\right| < \epsilon_n \mid \frac{m}{n} \in (a, b)\right) \to 1 \tag{B.2}$$

Proof. Firstly, we make the observation that if $f(k)$ satisfies above conditions, then we can find a positive ϵ small enough such that for $\forall k \in (k_0 + \epsilon, b)$, $f(k) < f(k_0 + \epsilon)$ holds.

From this, for $b > k = \frac{m}{n} > k_0 + \epsilon$

$$\ln(\frac{P_m}{P_{m-1}}) = f(k) < f(k_0 + \epsilon) \tag{B.3}$$

$$P_m < P_{m-1} \exp(f(k_0 + \epsilon)) \tag{B.4}$$

$$P_m < P_{m-l} \exp(f(k_0 + \epsilon))^l \tag{B.5}$$

$$\sum_{i=n(k_0+\epsilon)}^{nb} P_i < \sum_{i=n(k_0+\epsilon)}^{nb} P_{n(k_0+\epsilon)} \exp(f(k_0+\epsilon))^{i-n(k_0+\epsilon)} < \frac{P_{n(k_0+\epsilon)}}{1-\exp(f(k_0+\epsilon))} \tag{B.6}$$

$$\Pr(\frac{m}{n} - k_0 \geq \epsilon, \frac{m}{n} \in (a,b)) < \frac{P_{n(k_0+\epsilon)}}{1-\exp(f(k_0+\epsilon))} \tag{B.7}$$

Similar approach can be used to derive

$$\Pr(\frac{m}{n} - k_0 \leq -\epsilon, \frac{m}{n} \in (a,b)) < \frac{P_{n(k_0-\epsilon)}}{1-\exp(f(k_0-\epsilon))} \tag{B.8}$$

From the Riemann sum over $f(k)$, ($m_0 \triangleq nk_0$)

$$\ln P_{n(k_0+\epsilon)} - \ln P_{m_0} = \sum_{i=m_0+1}^{n(k_0+\epsilon)} \ln P_i - \ln P_{i-1} = \sum_{i=m_0+1}^{n(k_0+\epsilon)} f(i/n) \leq n \int_{k_0}^{k_0+\epsilon} f(k)dk \tag{B.9}$$

As a result, the following bound is derived:

$$\Pr(\frac{m}{n} - k_0 \geq \epsilon, \frac{m}{n} \in (a,b)) \leq \frac{P_{m_0} \exp(n \int_{k_0}^{k_0+\epsilon} f(k)dk)}{1-\exp(f(k_0+\epsilon))} \tag{B.10}$$

The above analysis can be repeated for the negative part to obtain the following bound:

$$\Pr(\frac{m}{n} - k_0 \leq -\epsilon, \frac{m}{n} \in (a,b)) < \frac{P_{m_0} \exp(n \int_{k_0-\epsilon}^{k_0} f(k)dk)}{1-\exp(f(k_0-\epsilon))} \tag{B.11}$$

Next, Taylor series expansion is used to linearize $f(k_0+\epsilon)$.

$$f(k_0+\epsilon) = f(k_0) + f'(k_0)\epsilon + o(\epsilon) \tag{B.12}$$

For small ϵ, $f(k_0+\epsilon) \approx f'(k_0)\epsilon$. The bound from (B.10) becomes,

$$\Pr(\frac{m}{n} - k_0 \geq \epsilon, \frac{m}{n} \in (a,b)) < P_{m_0} \frac{\exp(f'(k_0)n\epsilon^2/2)}{1-\exp(f'(k_0)\epsilon)} \tag{B.13}$$

We want to choose an ϵ_n sequence such that both the sequence and the above bound converges to 0. $\epsilon_n = cn^{-1/3}$ satisfies this condition since,

$$\lim_{n\to\infty} \frac{\exp(f'(k_0)n\epsilon^2/2)}{1-\exp(f'(k_0)\epsilon)} = \lim_{n\to\infty} \frac{\exp(c^2 f'(k_0)n^{1/3}/2)}{1-\exp(cf'(k_0)n^{-1/3})} = 0 \tag{B.14}$$

Similar arguments can be used for the negative side and sum of (B.11) and (B.13) gives the following.

$$\frac{\Pr\left(\left|\frac{m}{n}-k_0\right| \geq \epsilon_n, \frac{m}{n} \in (a,b)\right)\right)}{P_{m_0}} \to 0 \tag{B.15}$$

Then, since $\Pr(\frac{m}{n} \in (a,b)) \geq P_{m_0}$,

$$\Pr\left(\left|\frac{m}{n}-k_0\right| \geq \epsilon_n \mid k \in (a,b)\right) \to 0 \tag{B.16}$$

Finally, the equation in second property is obtained:

$$\Pr\left(\left|\frac{m}{n}-k_0\right| < \epsilon_n \mid k \in (a,b)\right) \to 1 \tag{B.17}$$

□

APPENDIX C

PROOF OF THEOREM 3

We only give the proof for the first part of the theorem. Second part follows similarly, by switching S_0 and k_0 with S_2 and k_2. Under the conditions given in part (i), we will first prove that

$$\Pr(S_0) \to 1, \Pr(S_1) \to 0, \Pr(S_2) \to 0 \tag{C.1}$$

To show that $\Pr(S_2) \to 0$, we use Lemma 11. Lemma 11 can be used for S_0 and S_2 regions since k_0 and k_2 satisfy the conditions of the Lemma over regions $\left(0, \frac{k_0+k_1}{2}\right)$ and $\left(\frac{k_1+k_2}{2}, 1\right)$ respectively. Using property (i) of Lemma 11,

$$\Pr(|\frac{m}{n} - k_2| \geq \epsilon_n, S_2) \leq P_{m_2} o(1) \tag{C.2}$$

Since P_{m_2} is the local maxima, we can use it as an upper bound over all P_m values in the region between $k_2 - \epsilon_n$ and $k_2 + \epsilon_n$, which will also be inside S_2.

$$\Pr(|\frac{m}{n} - k_2| < \epsilon_n, S_2) \leq P_{m_2} 2n\epsilon_n = P_{m_2} 2cn^{2/3} \tag{C.3}$$

$$\Pr(S_2) \leq P_{m_2}(2cn^{2/3} + o(1)) \tag{C.4}$$

Now we define k_3 such that $\int\limits_{k_3}^{k_2} f(k')dk' = 0$ and $k_3 \in (k_0, k_2)$ holds. Such k_3 exists since $\int\limits_{k_0}^{k_2} f(k')dk' < 0$ and $f(k)$ is continuous. Then,

$$\ln\left(\frac{P_{m_3}}{P_{m_2}}\right) \to n \int\limits_{k_3}^{k_2} f(k')dk' = 0 \tag{C.5}$$

P_{m_3} can be used as a lower bound in interval between k_0 and k_3, similar to how P_{m_2} was used as an upper bound. Furthermore, $f(k_3)$ must be negative and thus $k_3 \in (k_0, k_1)$. Hence, k_3 does not lie in the region S_2 and regions (k_0, k_3) and S_2 are disjoint:

$$1 - \Pr(S_2) \geq \Pr\left(\frac{m}{n} \in (k_0, k_3)\right) \geq P_{m_3} n(k_3 - k_0) \tag{C.6}$$

Ratio of (C.4) and (C.6) results in the following:

$$\frac{\Pr(S_2)}{1 - \Pr(S_2)} \leq \frac{P_{m_2}}{P_{m_3}} \left(\frac{c}{k_3 - k_0} n^{-1/3} + o(1/n)\right) \tag{C.7}$$

Upper bound of (C.7) goes to 0, so $\Pr(S_2)/(1-\Pr(S_2))$ goes to 0 as well. As a result, $\Pr(S_2) \to 0$. Next, we derive $\Pr(S_1)$. Region S_1 corresponds to the local minima or the valley of the PMF over the number of active sources. The point with maximum probability (in PMF) in S_1 will be one of the endpoints. We use this probability as an upper bound over S_1.

$$\Pr(S_1) < n(\frac{k_2 - k_0}{2}) max\{P_{n\frac{k_0+k_1}{2}}, P_{n\frac{k_1+k_2}{2}}\} \tag{C.8}$$

$$\ln\left(\frac{P_{n\frac{k_0+k_1}{2}}}{P_{nk_0}}\right) \to n \int_{k_0}^{\frac{k_0+k_1}{2}} f(k')dk' \tag{C.9}$$

$$\ln\left(\frac{P_{n\frac{k_1+k_2}{2}}}{P_{nk_2}}\right) \to -n \int_{\frac{k_1+k_2}{2}}^{k_2} f(k')dk' \tag{C.10}$$

Since $\int_{k_0}^{\frac{k_0+k_1}{2}} f(k')dk' < 0$ and $\int_{\frac{k_1+k_2}{2}}^{k_2} f(k')dk' > 0$, both $P_{n\frac{k_0+k_1}{2}}$ and $P_{n\frac{k_1+k_2}{2}}$ decay exponentially as n grows, hence $\Pr(S_1) \to 0$. Since $\Pr(S_0) + \Pr(S_1) + \Pr(S_2) = 1$, we finally obtain $\Pr(S_0) \to 1$. Following bound originates from the conditional probability:

$$\Pr(|\frac{m}{n} - k_0| < \epsilon_n) \geq \Pr(|\frac{m}{n} - k_0| < \epsilon_n | S_0) \Pr(S_0) \tag{C.11}$$

From property (ii) of Lemma 11,

$$\Pr(|\frac{m}{n} - k_0| < \epsilon_n | S_0) \to 1 \tag{C.12}$$

Finally, $\Pr(S_0) \to 1$ is used along with (C.11) and (C.12), to conclude the proof:

$$\Pr(|\frac{m}{n} - k_0| < \epsilon_n) \to 1 \tag{C.13}$$

APPENDIX D

PROOF OF THEOREM 8

The proof of this theorem follows from the proof of Theorem 1 with minor differences. The type structure of Theorem 1 is not changed and steady state properties of the pivoted MC remains identical. The results of Lemma 2 are modified due to the different throughput expression of MuMiSTA:

Lemma 12. *Let $S_1^{\boldsymbol{P}}$ and $S_2^{\boldsymbol{P}}$ be two arbitrary states in $\boldsymbol{P}^{\Gamma}$ where the state of the pivot source is equal for both states. Let the types of $S_1^{\boldsymbol{P}}$ and $S_2^{\boldsymbol{P}}$ be:*

$$\boldsymbol{T^P}\langle S_1^{\boldsymbol{P}}\rangle = (s, m_1, \{u_1, u_2, \ldots, u_{n-m_1-1}\})$$

$$\boldsymbol{T^P}\langle S_2^{\boldsymbol{P}}\rangle = (s, m_2, \{v_1, v_2, \ldots, v_{n-m_2-1}\})$$

i) Let $Q_1^{\boldsymbol{P}}$ be any state satisfying $\boldsymbol{T^P}\langle Q_1^{\boldsymbol{P}}\rangle = Q(S_1^{\boldsymbol{P}})$. Then,

$$\lim_{n\to\infty} \frac{\pi(S_1^{\boldsymbol{P}})}{\pi(Q_1^{\boldsymbol{P}})} = 1 \tag{D.1}$$

ii) If $m_1 = m_2$, then

$$\lim_{n\to\infty} \frac{\pi(S_1^{\boldsymbol{P}})}{\pi(S_2^{\boldsymbol{P}})} = 1 \tag{D.2}$$

iii) If $m_1 = m_2 + 1$, then

$$\lim_{n\to\infty} \frac{\pi(S_1^{\boldsymbol{P}})}{n\,\pi(S_2^{\boldsymbol{P}})} = k\left(\frac{1}{T(\alpha x, \tau_2, ..., \tau_W)} - 1\right) \tag{D.3}$$

where $\lim_{n\to\infty}\frac{m_1}{n} = k$ and $\lim_{n\to\infty}\tau n = \alpha$. ($k, \alpha \in \mathbb{R}^+$)

Rest of the proof follows accordingly.

APPENDIX E

PROOF OF PROPOSITION 6

E.1 First Part

For any $j \in \mathbb{Z}^+$, the following holds due to (4.41):

$$\frac{y_{j+1}}{y_j} = \frac{x_j p_{10} r_j + y_j p_{00}}{y_j} \geq p_{00} \tag{E.1}$$

The statement of the proposition follows almost immediately:

$$\frac{y_k}{y_l} = \prod_{j=l}^{k-1} \frac{y_{j+1}}{y_j} \geq \prod_{j=l}^{k-1} p_{00} = p_{00}^{k-l} \tag{E.2}$$

E.2 Second Part

For the case of $k \leq j$, the values of x_k and y_k are not influenced by r_j due to causality and therefore:

$$\frac{\mathrm{d}x_k}{\mathrm{d}r_j} = \frac{\mathrm{d}y_k}{\mathrm{d}r_j} = 0 \tag{E.3}$$

< For the case of $k = j + 1$; $\frac{\mathrm{d}x_k}{\mathrm{d}r_j} = \frac{\mathrm{d}(p_{11} r_j x_j + p_{01} y_j)}{\mathrm{d}r_j} = p_{11} x_j \geq 0$. Similarly, $\frac{\mathrm{d}y_k}{\mathrm{d}r_j} = p_{10} x_j \geq 0$. For the remaining cases, we use induction. Assume the proposition statement holds for $k = n$ case. For $k = n + 1$:

$$\frac{\mathrm{d}x_{n+1}}{\mathrm{d}r_j} = p_{11} r_n \frac{\mathrm{d}x_n}{\mathrm{d}r_j} + p_{01} \frac{\mathrm{d}y_n}{\mathrm{d}r_j} \geq 0, \tag{E.4}$$

$$\frac{\mathrm{d}y_{n+1}}{\mathrm{d}r_j} = p_{10} r_n \frac{\mathrm{d}x_n}{\mathrm{d}r_j} + p_{00} \frac{\mathrm{d}y_n}{\mathrm{d}r_j} \geq 0. \tag{E.5}$$

By mathematical induction, proof is complete.

$$\lambda \to \frac{p_{01}(p_{10} + p_{01})}{p_{01}(p_{10} + p_{01}) + p_{10}}\lambda \tag{E.6}$$